FLIGHT MECHANICS

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CHAPTER I

INTRODUCTORY CONCEPTS

1-1. Units

In the study of mechanics we are concerned with the interactions of material bodies. More specifically, we study the interactions and motions of an idealized system composed of particles, extended bodies, and fluids. By this means we are able to explain and to predict many of the mechanical phenomena that we can observe and measure.

In order to be able to distinguish between the various sorts of mechanical variables and to assign numerical values to their magnitudes, one is led to a consideration of units and their dimensions. For our purposes, three fundamental units are sufficient, namely, the units of length, force and time. All other units such as the units of mass, velocity, area, momentum, etc., can be expressed in terms of the fundamental units and are known as derived units.

We shall adopt the English gravitational system of units in which the unit of length is the foot, the unit of force is the pound, and the unit of time is the second.

Definitions of basic units. For our purposes, the definitions of the basic units can be clarified by thinking of operational procedures illustrating their meaning. For example, the foot could be defined in terms of the distance between two lines on a specific bar at a standard temperature. The pound as a unit of force could be defined in terms of the effort required to produce a given extension in a certain spring under standard conditions. The unit of time, the second, could be defined in terms of the period of the earth's rotation on its axis with respect to the fixed stars.

These definitions do not coincide with the standard definitions in actual use. In particular, the unit of mass is considered as fundamental in accordance with international agreement rather than the unit of force. In this case, the unit of force is a derived unit and is that force required to give the standard mass a certain acceleration.

Derived Units. We have seen that the gravitational system of units regards the units of length, force and time as fundamental and that all other quantities involved in the study of mechanics can be expressed in terms of these fundamental units. Perhaps the most important of these derived units is the unit of mass known as the slug. The slug has the units of lb sec^2/ft and is that mass which one pound of force accelerates at the rate of 1 ft/sec^2. Its weight, that is, the gravitational force acting on it, is mg or approximately 32.17 lbs. We will generally specify mass in lb sec^2/ft rather than in slugs in order to keep the number of units at a minimum. Note that we do not use the pound as a unit of mass in this presentation, even though it is the legal standard of mass in this country. We will always use the term "pound" as a unit of force.

Some of the most commonly used quantities are listed below with their corresponding gravitational units and dimensions. The dimensions in each case are designated by the algebraic terms in the brackets. F, L, and T refer to units of force, length, and time, respectively. The exponents are those of the corresponding units in the middle column. The dimensions of a given quantity are fixed so long as the general system of units is not changed, i. e., whether one measures length in feet or centimeters is immaterial in specifying the dimensions. On the other hand, if one should choose mass rather than force as a fundamental unit, the dimensions of many mechanical quantities would be changed.

Quantity	Gravitational Units	Dimensions
length	foot	[L]
force	pound	[F]
time	second	[T]
mass	lb sec^2/ft(slug)	[FT^2L^{-1}]
velocity	ft/sec	[LT^{-1}]
acceleration	ft/sec^2	[LT^{-2}]
energy (work)	ft lb	[FL]
angular velocity	1/sec	[T^{-1}]
moment	lb ft	[FL]
moment of inertia	lb ft sec^2	[FLT^2]
linear momentum	lb sec	[FT]
angular momentum	lb ft sec	[FLT]
linear impulse	lb sec	[FT]
angular impulse	lb ft sec	[FLT]

Dimensional homogeneity. It is helpful to carry along the units when performing numerical computations, treating them as algebraic quantities. This implies that any terms which are added or subtracted must have the same dimensions. Also, of course, the expressions on each side of an equality must have the same dimensions. Furthermore, any argument of a transcendental function such as the trigonometric functions, exponential functions, Bessel functions, etc., must be dimensionless, that is, all dimensional exponents must be zero. One will sometimes find apparent exceptions to this rule but in all cases some of the coefficients will prove to have unsuspected dimensions or else the equation is an empirical approximation not based on physical law. In checking for dimensional homogeneity one should note that the unit of angular displacement, the radian, is dimensionless.

Conversion of units. The algebraic manipulation of units often requires the conversion of one set of units to another set having the same dimensions. It is usually advisable to convert to three basic units such as feet, pounds, and seconds rather than carrying along derived units such as slugs or perhaps several different units of length. In this conversion process one does not change the magnitudes of any of the physical quantities, but only their form of expression.

A convenient method of changing units is to multiply by one or more fractions whose magnitude is unity but in which the numerator and denominator are expressed in different units. Suppose, for example, that one wishes to convert furlongs per fortnight into inches per second. One finds that

$$\begin{aligned} 1 \text{ furlong} &= 220 \text{ yd} \\ 1 \text{ yd} &= 36 \text{ in} \\ 1 \text{ fortnight} &= 14 \text{ days} \\ 1 \text{ day} &= 86{,}400 \text{ sec.} \end{aligned}$$

Therefore,

$$1 \frac{\text{furlong}}{\text{fortnight}} = \frac{1 \text{ furlong}}{1 \text{ fortnight}} \cdot \frac{1 \text{ fortnight}}{14 \text{ days}} \cdot \frac{1 \text{ day}}{86{,}400 \text{sec}} \cdot \frac{220 \text{ yd}}{1 \text{ furlong}} \cdot \frac{36 \text{ in}}{1 \text{ yd}}$$

$$= 6.548 \times 10^{-3} \frac{\text{in}}{\text{sec}}$$

1-2. Vectors

Scalars, vectors, tensors. In this treatment of mechanics, the vectorial approach will be emphasized rather than the variational approach. It is important, then, to have a firm grasp of basic vector operations that will be used in its development.

First let us distinguish among scalars, vectors, and other tensors of higher order. Briefly, a scalar quantity is expressible as a single number. A vector is expressible as a column of numbers. A second-order tensor (the inertia tensor, for example) can be written as a two-dimensional array of numbers. Similarly, a third-order tensor can be written as a three-dimensional array of numbers, and so on. Thus, one concludes that a vector is a first-order tensor and also that a scalar is a zero-order tensor.

We will not be concerned with tensors of order higher than two and therefore no more than a two-dimensional array of numbers will be needed to express the quantities encountered. This circumstance enables us to use matrix notation rather than the more general but less familiar tensor notation. Also, except for the chapter on vibration theory, we will be concerned with phenomena in a three-dimensional space and therefore each matrix or array will have no more than three rows or columns and each vector will have no more than three components.

Type of vectors. Instead of thinking of a vector as a one-dimensional array of numbers, we might consider its graphical representation as a line segment having a definite magnitude and direction. For such a vector in three-dimensional space we can identify its three components with the numbers of the array. This sort of vector quantity which has magnitude and direction but no specified location is known as a free vector. For example, the translational velocity of nonrotating body can be represented by a free vector, this vector specifying the velocity of any point in the body. Another example is a force vector when considering its effect upon translational motion.

On the other hand, when one considers the effect of a force on the rotational motion of a rigid body, not only the magnitude and direction of the force, but also its line of action are important. In this case the moment acting on the body will depend upon the line of action of the force but will be independent of the precise point of application along that line. A vector of this sort is known as a sliding vector.

The third principal type of vector is the bound vector. In this case, the magnitude, direction and point of application are specified. An example of a bound vector is a force acting on an elastic body, the elastic deformation being dependent upon the exact location of the force.

Most of the vectors that are encountered in mechanics are free vectors. But regardless of the types of vectors involved, the vector operations such as summation, multiplication, differentiation, etc., will be performed using only the properties of magnitude and direction. If the location is important, this will influence the statement of the vector operation but will not affect the procedure to be followed in its evaluation.

Unit vectors. When a vector is multiplied by a positive scalar, the resulting vector has the same direction but the magnitude is multiplied by the scalar factor. Conversely, when a vector is multiplied by a negative scalar, the direction is exactly reversed, but the magnitude is again changed by a factor equal to the magnitude of the scalar. This being the case, one can always think of a vector as being the product of a scalar equal to the magnitude of the vector and a vector of unit length pointing in a direction parallel to the vector. (See Figure 1)

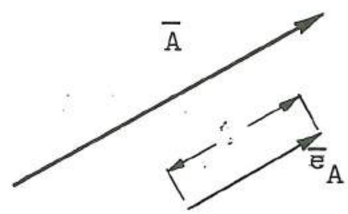

Figure 1

In other words we can write

$$\overline{A} = A\,\overline{e}_A \tag{1-1}$$

where the scalar factor A specifies the magnitude of vector $\overline{A}$ and the unit vector $\overline{e}_A$ specifies its direction.

Addition of vectors. The vectors $\overline{A}$ and $\overline{B}$ can be added as shown in Figure 2 to give the resultant vector $\overline{C}$. To add $\overline{B}$ to $\overline{A}$, translate B until its beginning is at the arrow of $\overline{A}$. The vector sum is indicated by the directed line segment from the beginning of $\overline{A}$ to the arrow of $\overline{B}$. The translation process is permissible because we are concerned only with the free vector properties of magnitude and direction and not the location.

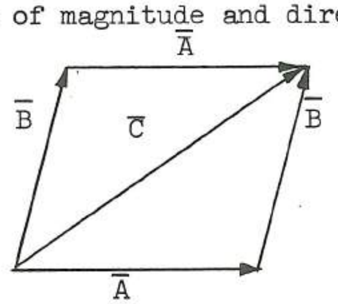

Figure 2

It can be seen that

$$\overline{C} = \overline{A} + \overline{B} = \overline{B} + \overline{A} \tag{1-2}$$

and therefore vector addition is commutative. By the same process as in Figure 2 we could add more than two vectors. For example, if we add another vector $\overline{D}$ to $\overline{C}$, we obtain

$$\overline{C} + \overline{D} = (\overline{A} + \overline{B}) + \overline{D} = \overline{E} \tag{1-3}$$

as shown in Figure 3.

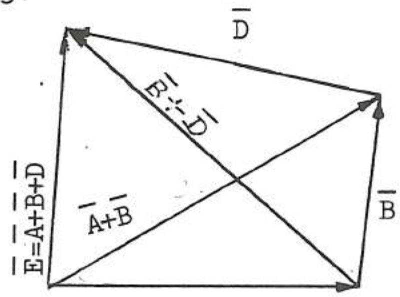

Figure 3

But we need not have grouped the vectors in this way. Again from Figure 3, we see that

$$\overline{E} = \overline{A} + (\overline{B} + \overline{D}) \tag{1-4}$$

From (1-3) and (1-4) we see that

$$(\overline{A} + \overline{B}) + \overline{D} = \overline{A} + (\overline{B} + \overline{D}) \tag{1-5}$$

illustrating that vector addition is associative.

In summary, because of the cummutative and associative properties of vector addition, we can dispense with the parentheses in a series of additions and perform the additions in any order.

Components of a vector. If a given vector $\overline{A}$ is equal to the sum of several vectors, these vectors can be considered as component vectors of $\overline{A}$. Since component vectors defined in this way are not unique, it is the usual practice to specify their directions, generally using three unit vectors. If we specify further that the unit vectors be mutually orthogonal, as in the Cartesian system of Figure 4, then the vector components are

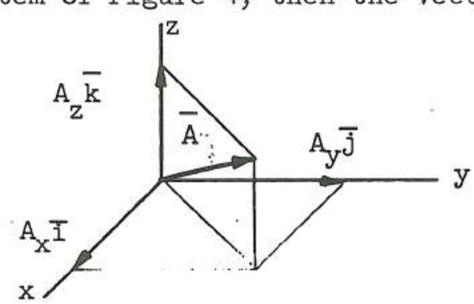

Figure 4

$A_x\overline{i}$, $A_y\overline{j}$ and $A_z\overline{k}$ where $\overline{i}$, $\overline{j}$, and $\overline{k}$ are unit vectors in the positive x, y, and z directions, respectively.

$$\overline{A} = A_x\overline{i} + A_y\overline{j} + A_z\overline{k} \qquad (1\text{-}6)$$

The magnitudes A_x, A_y, and A_z are known as the scalar components of $\overline{A}$ in the given directions. For this case where the unit vectors form an orthogonal set, the components A_x, A_y, and A_z are also equal to the orthogonal projections of the vector $\overline{A}$ on the x, y, and z axes, respectively.

The use of components can simplify vector operations. For example, if the vectors $\overline{A}$ and $\overline{B}$ are specified using the same set of unit vectors, then the components of the vector sum are merely the individual sums of the corresponding components. In other words, if

$$\overline{A} = A_1\,\overline{e}_1 + A_2\,\overline{e}_2 + A_3\,\overline{e}_3 \qquad (1\text{-}7)$$

and if

$$\overline{B} = B_1\,\overline{e}_1 + B_2\,\overline{e}_2 + B_3\,\overline{e}_3 \qquad (1\text{-}8)$$

then

$$\overline{A} + \overline{B} = (A_1 + B_1)\overline{e}_1 + (A_2 + B_2)\overline{e}_2 + (A_3 + B_3)\overline{e}_3 \qquad (1\text{-}9)$$

whether or not $\overline{e}_1$, $\overline{e}_2$, and $\overline{e}_3$ form an orthogonal set of unit vectors.

Scalar product. The scalar product of two vectors $\overline{A}$ and $\overline{B}$ is also known as the dot product and is given by

$$\overline{A}\cdot\overline{B} = A\,B\cos\theta \qquad (1\text{-}10)$$

It can be seen that

$$\overline{A}\cdot\overline{B} = \overline{B}\cdot\overline{A} \qquad (1\text{-}11)$$

so scalar multiplication is commutative.

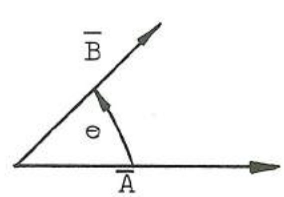

Figure 5

If the vectors $\overline{A}$ and $\overline{B}$ are given by Equations (1-7) and (1-8) where the unit vectors $\overline{e}_1$, $\overline{e}_2$, and $\overline{e}_3$ are mutually orthogonal, then

$$\overline{A}\cdot\overline{B} = A_1 B_1 + A_2B_2 + A_3B_3 \tag{1-12}$$

Additional terms are required for the case where $\overline{e}_1$, $\overline{e}_2$, and $\overline{e}_3$ are not orthogonal.

In obtaining Equation (1-12) we have used the fact that scalar multiplication is distributive, that is,

$$\overline{P}\cdot(\overline{Q} + \overline{R}) = \overline{P}\cdot\overline{Q} + \overline{P}\cdot\overline{R} \tag{1-13}$$

Vector product. Referring again to Figure 5 we can define the vector product or cross product as

$$\overline{A} \times \overline{B} = A\, B \sin \theta\, \overline{k} \tag{1-14}$$

where $\overline{k}$ is a unit vector perpendicular to and out of the page. In general the magnitude of the vector product is the product of the vector amplitudes times the sine of the angle traversed in going from the first vector to the second. The direction of the product vector is perpendicular to the plane of $\overline{A}$ and $\overline{B}$ (assuming the vectors are translated such that they have a common origin) and the sense is determined from the right-hand rule as A is rotated to B.

It can be seen that

$$\overline{A} \times \overline{B} = -\overline{B} \times \overline{A} \tag{1-15}$$

so the vector product is not commutative.

Furthermore, one can show that the vector product is not associative.

$$(\overline{A} \times \overline{B}) \times \overline{C} \neq \overline{A} \times (\overline{B} \times \overline{C}) \tag{1-16}$$

However, it is distributive

$$\overline{A} \times (\overline{B} + \overline{C}) = (\overline{A} \times \overline{B}) + (\overline{A} \times \overline{C}) \tag{1-17}$$

It should be pointed out that the vector triple product $\overline{A} \times (\overline{B}\times\overline{C})$ lies in the plane determined by $\overline{B}$ and $\overline{C}$.

$$\overline{A} \times (\overline{B} \times \overline{C}) = (\overline{A}\cdot\overline{C})\overline{B} - (\overline{A}\cdot\overline{B})\, \overline{C} \tag{1-18}$$

scalar, whereas the vector $\overline{A} + \Delta\overline{A}$ corresponds to $u + \Delta u$. Then the derivative is found by the limiting process

$$\frac{d\overline{A}}{du} = \lim_{u \to 0} \frac{\Delta\overline{A}}{\Delta u} \tag{1-22}$$

For the case where the vector is differentiated with respect to time, geometrical significance can be attached to the derivative. It is simply the velocity of the point of the vector when the other end is fixed.

Differentiation of a vector is a distributive operation, that is,

$$\frac{d}{du}(\overline{A} + \overline{B}) = \frac{d\overline{A}}{du} + \frac{d\overline{B}}{du} \tag{1-23}$$

So one can obtain the derivative of a vector as the sum of the derivatives of its components. If $\overline{A}$ is given by equation (1-6) then,

$$\frac{d\overline{A}}{du} = \frac{dA_x}{du}\,\overline{i} + \frac{dA_y}{du}\,\overline{j} + \frac{dA_z}{du}\,\overline{k} \tag{1-24}$$

where the Cartesian unit vectors $\overline{i}$, $\overline{j}$, and $\overline{k}$ are assumed to be independent of u.

Often one encounters time derivatives of vectors expressed in terms of unit vectors whose direction varies with time. If

$$\overline{A} = A_1\,\overline{e}_1 + A_2\,\overline{e}_2 + A_3\,\overline{e}_3 \tag{1-25}$$

then

$$\frac{d\overline{A}}{dt} = \dot{A}_1\,\overline{e}_1 + \dot{A}_2\,\overline{e}_2 + \dot{A}_3\,\overline{e}_3 + A_1\,\dot{\overline{e}}_1 + A_2\,\dot{\overline{e}}_2 + A_3\,\dot{\overline{e}}_3 \tag{1-26}$$

1-3. Newton's Laws of Motion for a Particle

In 1687, Sir Isaac Newton stated the laws on which classical mechanics is based. These laws, referring the motion of a particle (i. e., a mass concentrated at a point) under the action of an external force, are as follows:

1. A particle under the action of no forces remains at rest or moves in a straight line with constant speed.

2. The rate of change of momentum is proportional to the impressed force and is in the direction in which the force acts.

3. To every action there is an equal and opposite reaction.

The law of motion. Newton's first two laws may be summarized by the equation

$$\overline{F} = k \frac{d}{dt} (m \overline{v}) = km\overline{a} \tag{1-27}$$

where m is the mass of the particle, $\overline{a}$ is its acceleration, $\overline{F}$ is the applied external force and where k is a positive constant whole value depends upon the choice of units. Note that the mass of a particle is constant in Newtonian mechanics; variable-mass systems are treated as a collection of particles. Also note that forces due to gravitational or electromagnetic fields are regarded as external forces.

Because of the fundamental nature of this equation, the units are chosen such that k = 1. For example, in the gravitational system the unit of mass, the slug (lb sec^2/ft), is that mass which is given an acceleration of 1 ft/sec^2 by a force of 1 lb. Similarly, in another system such as the cgs (centimetergram-second) system, the unit of force, the dyne, is chosen such that the constant k = 1.

So, with a proper choice of units, Equation (1-27) can be simplified to the form

$$\overline{F} = m \overline{a} \tag{1-28}$$

The question immediately arises as to a proper frame of reference with respect to which the acceleration is to be measured. Any reference frame in which Equation (1-28) applies is known as an inertial or Newtonian system. An example of such a system is one that is fixed with respect to the average position of the "fixed" stars. Another system which can be considered as inertial is a nonrotating system that is fixed with respect to center of the sun. In fact, for many engineering applications, a system fixed in the earth is satisfactory.

Now it can be shown that any system that is not rotating and is translating at a uniform velocity with respect to an inertial system is itself an inertial system. For example, if system B is translating at constant velocity $\overline{v}_{rel}$ with respect to an inertial system A, then denoting the velocity of a particle as viewed by observers on A and B by $\overline{v}_A$ and $\overline{v}_B$, respectively, we see that

$$\overline{v}_A = \overline{v}_B + \overline{v}_{rel} \tag{1-29}$$

Differentiating with respect to time and noting that the derivative of $\overline{v}_{rel}$ is zero, we obtain

$$\overline{a}_A = \overline{a}_B \tag{1-30}$$

where $\overline{a}_A$ and $\overline{a}_B$ are the accelerations of the particle as viewed from systems A and B respectively. So from the Newtonian, nonrelativistic, point of view, observers on systems A and B see identical forces, masses and accelerations and, therefore, Equation (1-28) is equally valid for each observer.

The law of action and reaction. Newton's third law is essentially a statement concerning the interaction forces between two particles. Its meaning can be clarified somewhat by writing it in the following form:

> When two particles exert forces on each other, these interaction forces are equal in magnitude, opposite in sense, and directed along the straight line joining the particles.

In other words, the force exerted by particle A on particle B is equal, opposite, and collinear to the force exerted by particle B on particle A. This added requirement of collinearity will be found to be essential for the conservation of angular momentum of an isolated mechanical system. The possibility of one particle exerting a moment on another particle does not arise because a point mass has no rotational inertia and thus cannot, by itself, exert a moment.

The law of addition of forces. Newton's laws do not specifically consider the case where a single particle is being acted upon simultaneously by two or more forces. In order to cover that possibility the following law can be stated:

> Two forces $\overline{P}$ and $\overline{Q}$ acting simultaneously on a particle are together equivalent to a single force $\overline{F} = \overline{P} + \overline{Q}$.

By similar reasoning we can conclude that the simultaneous action of two or more forces on a particle produces the same motion as a single force equal to their vector sum.

A further conclusion reached from the law of addition of forces and the law of motion is that the acceleration produced at a given time by the simultaneous action of several forces is equal to the vector sum of the accelerations produced by the individual forces acting separately. As a result, we can write a set of scalar equations which are together equivalent to the single vector equation in stating the law of motion. As an example, we can write for the case of a particle in a Cartesian inertial system:

$$F_x = m\ddot{x}$$
$$F_y = m\ddot{y} \qquad (1\text{-}31)$$
$$F_z = m\ddot{z}$$

where the total external force is

$$\overline{F} = F_x\,\overline{i} + F_y\,\overline{j} + F_z\,\overline{k} \qquad (1\text{-}32)$$

and the total acceleration is

$$\overline{a} = \ddot{x}\,\overline{i} + \ddot{y}\,\overline{j} + \ddot{z}\,\overline{k} \qquad (1\text{-}33)$$

D'Alembert's principle. Newton's law of motion given by Equation (1-28) could have been written in the form

$$\overline{F} - m\overline{a} = 0 \qquad (1\text{-}34)$$

If we consider the term $- m\,\overline{a}$ to be another force, known as the "inertial force" or the "reversed effective force", then Equation (1-34) can be considered as an equation of statics and the methods of statics can be applied to the system. For some problems, this slightly different viewpoint is

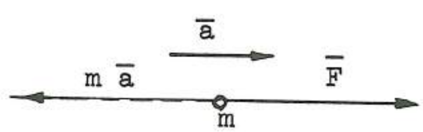

Figure 8

helpful, but care must be taken in its use. In particular, one must be careful not to confuse inertial forces with the real external forces applied to the particle. We will always designate real forces by a solid arrow and inertial forces by a dashed arrow.

Another approach is to think of the inertial force as a reaction force exerted by the particle on the outside world in accordance with the law of action and reaction. This viewpoint is convenient when the motion of a particle is given and one desires to calculate the force it exerts on the remainder of the system. An example of this sort will be given in Chapter 2.

CHAPTER 2

KINEMATICS OF A PARTICLE

Kinematics is the study of the motions of particles and rigid bodies, disregarding the forces associated with these motions. It is purely geometrical or mathematical in nature and does not involve any physical laws such as Newton's laws.

In this chapter we will be concerned primarily with the kinematics of a particle, that is, with the motion of a point. Depending upon the circumstances we may at times choose to consider the point as being attached to a rigid body and at other times as being the location of an individual particle. In any event, we will be interested in calculating such quantities as the position, velocity, and acceleration vectors of the point with respect to various reference frames.

From the viewpoint of kinematics, there are no preferred frames of reference and, therefore, we might expect the general equations to reflect this property. On the other hand, the principal application of these equations will be the determination of motions with respect to an inertial frame. For this reason some of the following developments will be made in a more restricted sense than the nature of the subject requires.

2-1. Angular Velocity

The meaning of angular velocity. In the study of the kinematics of a particle one is frequently faced with the calculation of time derivatives of vector quantities, and, as we have seen in Equation (1-26), this may involve the rates of change of unit vectors, i.e., their rotation in space. Also, these vector quantities may be viewed from various reference frames that are in relative rotational motion. So it is important to understand the nature of these motions and, in particular, to understand the meaning of angular velocity.

First, note that the term "angular velocity" implies a reference frame from which the angular velocity is measured. In general, measurements of the angular velocity of an object made from different frames of reference produce different results. Second, angular velocity refers to the motion of a rigid body, or, in essence, to the motion of a set of three regidly connected points that are not collinear. The term has no unique meaning for a point or a straight line (or a vector) when one thinks in terms of three-dimensional space.

Since angular velocity is concerned with the motion of rigid bodies, it will be helpful to borrow some of the results that will be explained more fully when the kinematics of rigid bodies is discussed in Chapter 7. Let us consider the general motion of a rigid body with respect to a given reference frame. The velocities of all points in the rigid body are known if one specifies the velocity of a single point, called the base point, and the angular velocity of the body about that point. In other words, if one analyzes an infinitesimal displacement occurring during an interval Δt, it can be considered as the superposition of two separate displacements: (1) a translational displacement $\overline{\Delta s}$ (all points in the body having identical displacements along parallel lines), plus (2) a rotational displacement $\overline{\Delta\theta}$ about an axis through the base point, the base point being considered as fixed. The infinitesimal angular rotation $\overline{\Delta\theta}$ is a vector whose magnitude is the angle of rotation and whose direction is along an axis determined by those points not displaced by the infinitesimal rotation. The sense is in accordance with the right hand rule. The angular velocity is the vector

$$\overline{\omega} = \lim_{\Delta t \to 0} \frac{\overline{\Delta\theta}}{\Delta t} \tag{2-1}$$

Now if we should analyze the same infinitesimal displacement of the body, but choose a different base point, we would find that only the translational part of the motion would be changed; the infinitesimal rotation would be identical. Thus we can see that the angular velocity is a property of the body as a whole and is not dependent upon the choice of a base point. Therefore, the angular velocity $\overline{\omega}$ is a free vector.

Rigid body rotation about a fixed point. In order to concentrate upon the rotational aspects of the motion, assume that the base point of the rigid body is fixed at the origin O of a Cartesian system. Let us calculate the velocity $\overline{v}$ relative to the xyz system of a point P fixed in the body. For simplicity we could think of the xyz system as being fixed in inertial space, in which case $\overline{v}$ would be the absolute velocity of P and $\overline{\omega}$ would be the absolute angular rotation rate of the body.

Consider first the case where rotation takes place about a fixed axis, corresponding to a fixed direction of $\overline{\omega}$, as shown in Figure 2-1. The path of point P in this case is a circle of radius $r \sin \theta$, and its speed[1] is given by

$$\dot{s} = \omega r \sin \theta \tag{2-2}$$

where s is the displacement along the path of P, ω is the magnitude

1. Speed is the magnitude of the velocity and therefore is a scalar quantity. Sometimes, however, the term "velocity" is used in this sense.

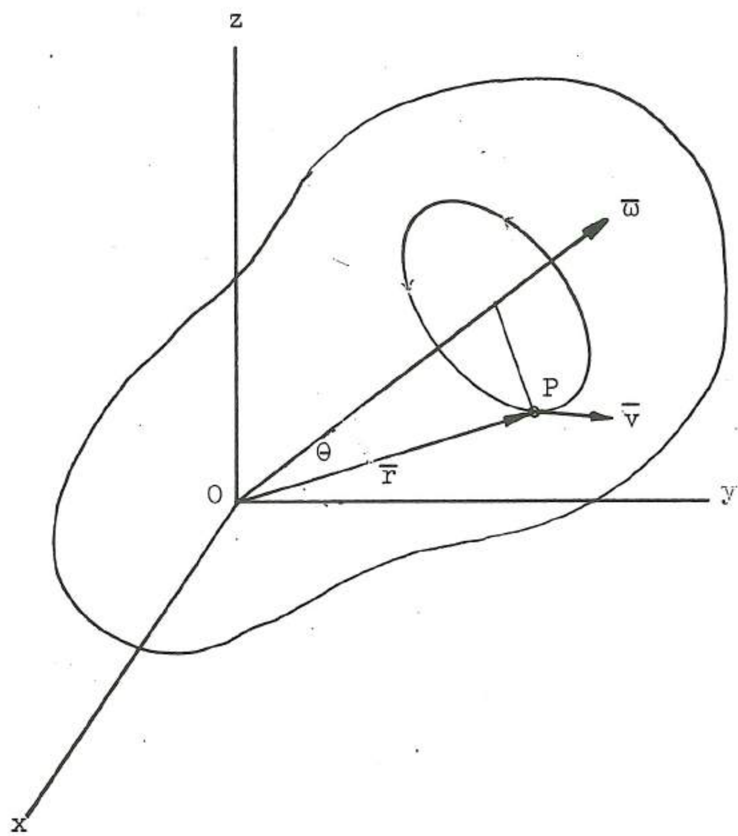

Figure 2-1

of $\overline{\omega}$, r is the constant length of the position vector drawn to P from O, and θ is the angle between $\overline{\omega}$ and $\overline{r}$. The velocity of P is

$$\overline{v} = \overline{\omega} \times \overline{r} \tag{2-3}$$

Equation (2-3) is also valid for the case where $\overline{\omega}$ is changing in magnitude or direction at a finite rate. In other words, the displacement of a point in the body during an infinitesimal interval is not affected by changes in $\overline{\omega}$ during that interval.

2-2. Time Derivative of a Unit Vector

We have obtained the rate of change of the position vector of a point P in a rigid body which is rotating about a fixed reference point O. A similar situation exists in the calculation of the rate of change of unit vectors. As was the case with the position vector to point P, the unit vectors are each of constant length. Furthermore, we have seen that the time derivative of a vector can be interpreted as the velocity of the point of the vector when the other end is fixed. So let us calculate the velocities of the unit vectors $\overline{e}_1$, $\overline{e}_2$, and $\overline{e}_3$ drawn from the origin of the fixed system X Y Z and rotating together as a rigid body with absolute angular velocity $\overline{\omega}$. (See Figure 2-2).

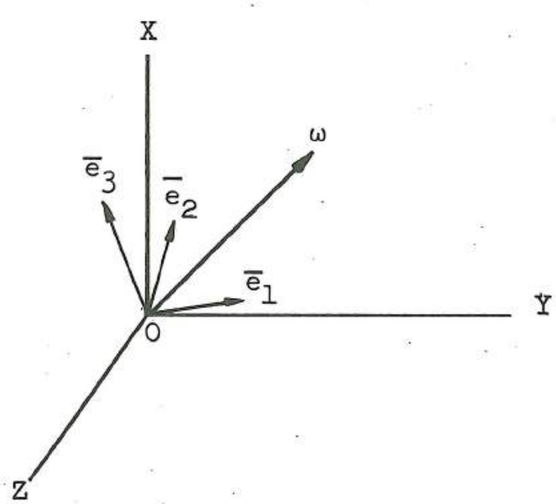

Figure 2-2

From Equation (2-3) we see that the velocities of the tips of the unit vectors, i.e., their time rates of change are

$$\begin{aligned} \dot{\overline{e}}_1 &= \overline{\omega} \times \overline{e}_1 \\ \dot{\overline{e}}_2 &= \overline{\omega} \times \overline{e}_2 \\ \dot{\overline{e}}_3 &= \overline{\omega} \times \overline{e}_3 \end{aligned} \tag{2-4}$$

As an example, let us calculate the rates of change of a **Cartesian** set of unit vectors $\overline{i}$, $\overline{j}$, $\overline{k}$ which are rotating with angular **velocity** $\overline{\omega}$ relative to X Y Z. From Equation (2-4), setting $\overline{e}_1 = \overline{i}$, $\overline{e}_2 = \overline{j}$, and $\overline{e}_3 = \overline{k}$, we find that

$$\dot{\overline{i}} = \overline{\omega} \times \overline{i}$$
$$\dot{\overline{j}} = \overline{\omega} \times \overline{j} \qquad (2\text{-}5)$$
$$\dot{\overline{k}} = \overline{\omega} \times \overline{k}$$

For the case where the vectors are expressed in terms of orthogonal components we can write the vector product in the form of a determinant. Thus if

$$\overline{A} = A_1\, \overline{e}_1 + A_2\, \overline{e}_2 + A_3\, \overline{e}_3 \qquad (2\text{-}6)$$

and

$$\overline{B} = B_1\, \overline{e}_1 + B_2\, \overline{e}_2 + B_3\, \overline{e}_3 \qquad (2\text{-}7)$$

and if the unit vectors form an orthogonal triad, then

$$\overline{A} \times \overline{B} = \begin{vmatrix} \overline{e}_1 & \overline{e}_2 & \overline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \qquad (2\text{-}8)$$

Applying Equation (2-8) to the case at hand, as given in Equation (2-5), and noting that

$$\overline{\omega} = \omega_x\, \overline{i} + \omega_y\, \overline{j} + \omega_z\, \overline{k} \qquad (2\text{-}9)$$

we obtain

$$\dot{\overline{i}} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix} = \omega_z\, \overline{j} - \omega_y\, \overline{k} \qquad (2\text{-}10)$$

Similarly,

$$\dot{\overline{j}} = \omega_x\, \overline{k} - \omega_x\, \overline{i} \qquad (2\text{-}11)$$

and

$$\dot{\bar{k}} = \omega_y \, \bar{i} - \omega_x \, \bar{j} \tag{2-12}$$

In each case the time derivative of the unit vector lies in a plane perpendicular to the vector, which follows directly from the definition of a vector cross product.

It is important to observe that we have in each case calculated the rate of change of the unit vector with respect to a fixed coordinate system X Y Z but have expressed the result in terms of the unit vectors of the moving system. This sort of approach will be used extensively in our later work, and the terminology involved should be made clear. The terms relative to or with respect to a given system mean as viewed by an observer on that system. On the other hand, the term referred to a certain system means that the vector is expressed in terms of the unit vectors of that system. For example, the absolute acceleration of a certain particle can be expressed in terms of the unit vectors of a fixed or of a moving coordinate system; in either event we are considering the same vector. However, the acceleration of the particle relative to a fixed and relative to a moving coordinate system would be in general quite different.

2-3. Velocity and Acceleration of a Particle in Several Coordinate Systems

Cartesian coordinates. Suppose that the position of the particle P relative to the x y z system is given by the vector

$$\bar{r} = x \, \bar{i} + y \, \bar{j} + z \, \bar{k} \tag{2-13}$$

Differentiation with respect to time gives

$$\bar{v} = \dot{\bar{r}} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x\dot{\bar{i}} + y\dot{\bar{j}} + z\dot{\bar{k}}$$

or

$$\bar{v} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} \tag{2-14}$$

since the unit vectors are constant in magnitude and direction, and thus, their time derivatives are zero. Differentiating again with respect to time, we obtain

$$\bar{a} = \ddot{\bar{r}} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k} \tag{2-15}$$

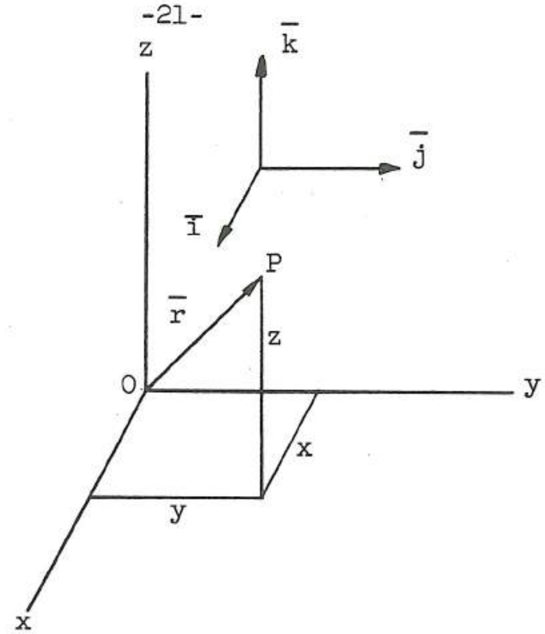

Figure 2-3

Cylindrical coordinates. In this case the position vector of P is

$$\bar{r} = r\bar{e}_r + z\bar{e}_z \tag{2-16}$$

Differentiating with respect to time, we obtain

$$\bar{v} = \dot{\bar{r}} = \dot{r}\bar{e}_r + \dot{z}\bar{e}_z + r\dot{\bar{e}}_r + z\dot{\bar{e}}_z \tag{2-17}$$

It can be seen that the unit vector triad does not retain the same orientation in space as the particle moves. It rotates with angular velocity

$$\bar{\omega} = \dot{\phi}\ \bar{e}_z \tag{2-18}$$

Therefore, we obtain from Equation (2-4) that

$$\begin{aligned} \dot{\bar{e}}_r &= \dot{\phi}\ \bar{e}_\phi \\ \dot{\bar{e}}_\phi &= -\dot{\phi}\ \bar{e}_r \\ \dot{\bar{e}}_z &= 0 \end{aligned} \tag{2-19}$$

resulting in

$$\bar{v} = \dot{r}\bar{e}_r + r\dot{\phi}\ \bar{e}_\phi + \dot{z}\bar{e}_z \tag{2-20}$$

Another differentiation with respect to time results in

$$\bar{a} = \ddot{\bar{r}} = \ddot{r}\bar{e}_r + (\dot{r}\ \dot{\phi} + r\ \ddot{\phi})\bar{e}_\phi + \ddot{z}\bar{e}_z + \dot{r}\dot{\bar{e}}_r + r\ \dot{\phi}\dot{\bar{e}}_\phi$$

which can be simplified with the aid of Equation (2-19) to give

$$\overline{a} = (\ddot{r} - r\,\dot{\phi}^2)\overline{e}_r + (r\,\ddot{\phi} + 2\dot{r}\dot{\phi})\overline{e}_\phi + \ddot{z}\,\overline{e}_z \qquad (2\text{-}21)$$

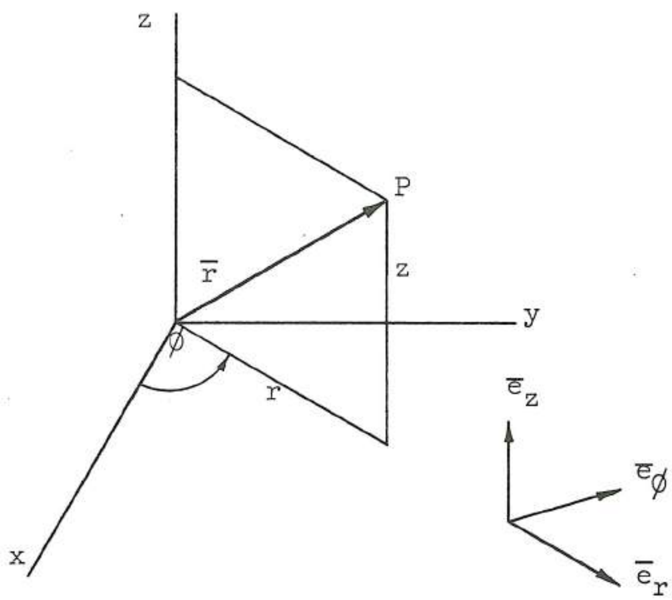

Figure 2-4

Spherical coordinates. The position vector in spherical coordinates is simply

$$\overline{r} = r\,\overline{e}_r \qquad (2\text{-}22)$$

The angular velocity of the unit vector triad is

$$\overline{\omega} = \dot{\phi}\,\overline{e}_z + \dot{\theta}\,\overline{e}_\phi \qquad (2\text{-}23)$$

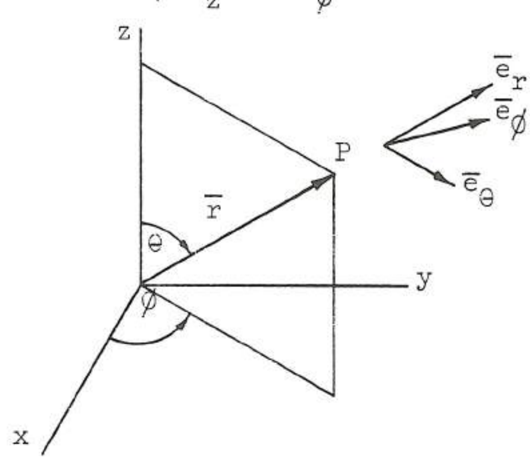

Figure 2-5

where $\overline{e}_z$ is a unit vector in the positive z direction. Taking components of $\dot{\phi}\,\overline{e}_z$ in the $\overline{e}_r$ and $\overline{e}_\theta$ directions, we obtain

$$\overline{\omega} = \dot{\phi} \cos\theta\, \overline{e}_r - \dot{\phi} \sin\theta\, \overline{e}_\theta + \dot{\theta}\, \overline{e}_\phi \tag{2-24}$$

Now, the unit vectors are orthogonal so we can use the determinant form for obtaining the cross product.

$$\dot{\overline{e}}_r = \overline{\omega} \times \overline{e}_r = \begin{vmatrix} \overline{e}_r & \overline{e}_\theta & \overline{e}_\phi \\ \dot{\phi}\cos\theta & -\dot{\phi}\sin\theta & \dot{\theta} \\ 1 & 0 & 0 \end{vmatrix}$$

or

$$\dot{\overline{e}}_r = \dot{\theta}\, \overline{e}_\theta + \dot{\phi} \sin\theta\, \overline{e}_\phi \tag{2-25}$$

By a similar process

$$\dot{\overline{e}}_\theta = - \dot{\theta}\, \overline{e}_r + \dot{\phi} \cos\theta\, \overline{e}_\phi \tag{2-26}$$

$$\dot{\overline{e}}_\phi = - \dot{\phi} \sin\theta\, \overline{e}_r - \dot{\phi} \cos\theta\, \overline{e}_\theta \tag{2-27}$$

So now we can evaluate the velocity

$$\overline{v} = \dot{\overline{r}} = \dot{r}\, \overline{e}_r + r\, \dot{\overline{e}}_r$$

or

$$\overline{v} = \dot{r}\, \overline{e}_r + r\, \dot{\theta}\, \overline{e}_\theta + r\, \dot{\phi} \sin\theta\, \overline{e}_\phi \tag{2-28}$$

Differentiating again with respect to time and substituting from Equations (2-25) through (2-27) for the derivatives of the unit vectors we obtain

$$\begin{aligned} \overline{a} = {} & (\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\phi}^2 \sin^2\theta)\,\overline{e}_r \\ & + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} - r\,\dot{\phi}^2 \sin\theta \cos\theta)\overline{e}_\theta \\ & + (r\,\ddot{\phi} \sin\theta + 2\dot{r}\,\dot{\phi} \sin\theta + 2\,r\,\dot{\theta}\,\dot{\phi} \cos\theta)\overline{e}_\phi \end{aligned} \tag{2-29}$$

<u>Tangential and normal components</u>. The velocity and acceleration of a particle P as it travels on a curved path through space may be

expressed in terms of tangential and normal unit vectors. Let us assume that the position of the particle is given by the distance s along the path from a reference point. At any moment, we can calculate the velocity and accelaration of P by considering it to be traveling on a circular arc whose center is at the instantaneous center of curvature C and whose radius ρ is equal to the radius of curvature. Then the velocity is given by

$$\bar{v} = \dot{s}\,\bar{e}_t \qquad (2\text{-}30)$$

where $\bar{e}_t$ is a unit vector tangent to the path at P and in the direction of the motion.

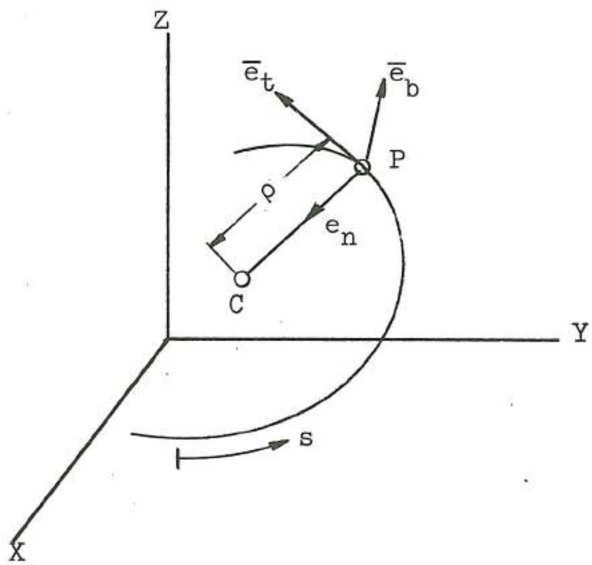

Figure 2-6

Let us define a second unit vector $\bar{e}_n$ in the direction of the line drawn from P to the instantaneous center of curvature C, i.e., in the direction of the principal normal. The third member of the unit vector triad is given by

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n \qquad (2\text{-}31)$$

and points in the direction of the binormal of the curve.

The unit vector triad rotates with an angular velocity

$$\bar{\omega} = \omega_t\,\bar{e}_t + \omega_b\,\bar{e}_b \qquad (2\text{-}32)$$

There is no normal component of $\bar{\omega}$ because $\bar{e}_n$ is defined to lie in the osculating plane, that is, in the plane of an infinitesimal arc at P.

Now, we can calculate the rate of change of $\bar{e}_t$.

$$\dot{\bar{e}}_t = \bar{\omega} \times \bar{e}_t = \omega_b \, \bar{e}_n \tag{2-33}$$

where

$$\omega_b = \frac{\dot{s}}{\rho} \tag{2-34}$$

giving

$$\dot{\bar{e}}_t = \frac{\dot{s}}{\rho} \bar{e}_n \tag{2-35}$$

Finally, the acceleration of P is found by differentiating Equation (2-30) with respect to time and making use of Equation (2-35).

$$\bar{a} = \ddot{s} \, \bar{e}_t + \dot{s} \, \dot{\bar{e}}_t = \ddot{s} \, \bar{e}_t + \frac{\dot{s}^2}{\rho} \bar{e}_n \tag{2-36}$$

2-4. Velocity and Acceleration of a Point in a Rigid Body

In Section 2-1 we calculated the absolute velocity of a point in a rigid body that is rotating about a fixed base point. Now consider the case where the base point A has a velocity $\bar{v}_A$ with respect to the inertial system XYZ and also is rotating with angular velocity $\bar{\omega}$ relative to this sytem. The absolute velocity of point P is

$$\bar{v} = \bar{v}_A + \bar{v}_{PA} \tag{2-37}$$

where $\bar{v}_{PA}$ is the velocity of P relative to A as viewed by an observer fixed in the XYZ system.

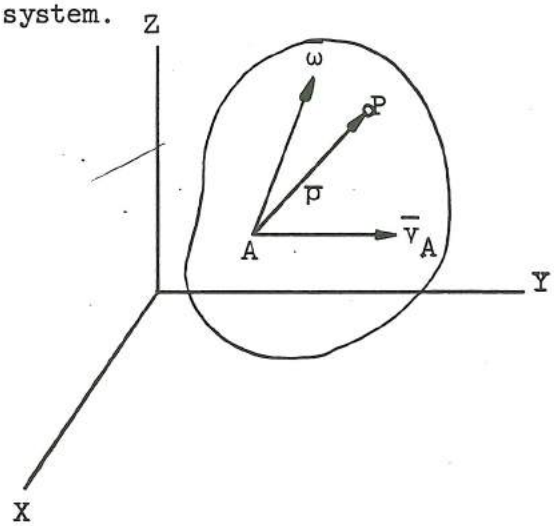

Figure 2-7

It is important to realize that the velocity of P relative to A will be different when viewed from various reference frames in relative rotational motion. Therefore, a statement of the relative velocity of two points should also specify the reference frame. An inertial frame will be assumed if none is stated explicitly.

The velocity $\overline{v}_{PA}$ can also be considered as the velocity of P as seen by an observer on a nonrotating system that is translating with A. Thus, we can use Equation (2-3) to obtain

$$\overline{v}_{PA} = \overline{\omega} \times \overline{\rho} \tag{2-38}$$

which combines with Equation (2-37) to give

$$\overline{v} = \overline{v}_A + \overline{\omega} \times \overline{\rho} \tag{2-39}$$

The acceleration of P is obtained by differentiating Equation (2-38) with respect to time.

$$\dot{\overline{v}} = \dot{\overline{v}}_A + \dot{\overline{\omega}} \times \overline{\rho} + \overline{\omega} \times \dot{\overline{\rho}} \tag{2-40}$$

Since $\overline{\rho}$ is a vector of constant magnitude and is fixed in the body, its derivative is

$$\dot{\overline{\rho}} = \overline{\omega} \times \overline{\rho} \tag{2-41}$$

Substituting Equation (2-41) into Equation (2-40) we obtain the absolute acceleration of P.

$$\overline{a} = \overline{a}_A + \dot{\overline{\omega}} \times \overline{\rho} + \overline{\omega} \times (\overline{\omega} \times \overline{\rho}) \tag{2-42}$$

2-5. Vector Derivatives in Rotating Systems

Suppose that a vector $\overline{A}$ is viewed by an observer on a fixed system XYZ and also by another observer on a moving system designated by the unit vector triad $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$, which is rotating with an angular velocity $\overline{\omega}$ relative to XYZ. No generality is lost by taking a common origin O.

At any instant, each observer might express the vector A in terms of the unit vectors of his system, and thus each would give a different set of components. Nevertheless, they would be viewing the same vector and a simple coordinate conversion based upon the relative orientation of the coordinate systems would provide a check of one observation with the other.

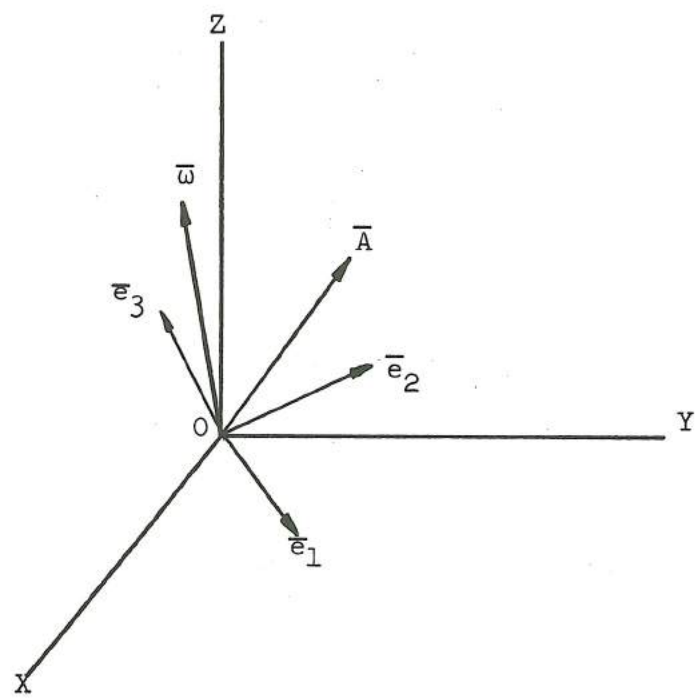
Z
$\overline{\omega}$
$\overline{A}$
$\overline{e}_3$
$\overline{e}_2$
O
Y
$\overline{e}_1$
X

Figure 2-8

But if each observer were to calculate the time rate of change of $\overline{A}$, the results would, in general, not agree, even after performing the coordinate conversion used previously. To clarify this point, recall from Equation (1-26) that the rate of change of $\overline{A}$ relative to the fixed system (but written in terms of the unit vectors of the rotating system) is

$$\dot{\overline{A}} = \dot{A}_1 \overline{e}_1 + \dot{A}_2 \overline{e}_2 + \dot{A}_3 \overline{e}_3 + A_1 \dot{\overline{e}}_1 + A_2 \dot{\overline{e}}_2 + A_3 \dot{\overline{e}}_3 \quad (2\text{-}43)$$

However, the rate of change of $\overline{A}$ relative to the rotating system is

$$(\dot{\overline{A}})_r = \dot{A}_1 \overline{e}_1 + \dot{A}_2 \overline{e}_2 + \dot{A}_3 \overline{e}_3 \quad (2\text{-}44)$$

since the unit vectors are fixed in this sytem. Using Equation (2-4) and remembering that multiplication of a vector is distributive, we obtain

$$A_1 \dot{\overline{e}}_1 + A_2 \dot{\overline{e}}_2 + A_3 \dot{\overline{e}}_3 = \overline{\omega} \times \overline{A} \quad (2\text{-}45)$$

Therefore, from Equations (2-43) through (2-45), we find that the absolute rate of change of A can be expressed in terms of its value relative to a rotating system as follows:

$$\dot{\overline{A}} = (\dot{\overline{A}})_r + \overline{\omega} \times \overline{A} \quad (2\text{-}46)$$

where $\overline{\omega}$ is the absolute angular velocity of the rotating system.

Since the result of Equation (2-46) is based upon kinematics alone, and not upon physical law, we need not consider either system as being more fundamental than the other. Therefore, if we call them system A and system B, respectively, we can write

$$(\dot{\overline{A}})_A = (\dot{\overline{A}})_B + \overline{\omega}_{BA} \times \overline{A} \quad (2\text{-}47)$$

where $\overline{\omega}_{BA}$ is the rotation rate of system B as viewed from system A. Since the result must be symmetrical with respect to the two systems, we could also write

$$(\dot{\overline{A}})_B = (\dot{\overline{A}})_A + \overline{\omega}_{AB} \times \overline{A} \quad (2\text{-}48)$$

where, of course,

$$\overline{\omega}_{AB} = -\overline{\omega}_{BA} \quad (2\text{-}49)$$

2-6. Motion of a Particle in a Moving Coordinate System.

Now we will use the general result of Equation (2-46) to obtain the equations for the absolute velocity and acceleration of a particle P that is in motion relative to a moving coordinate system.

The system XYZ is an inertial system. The xyz system is rotating with with absolute angular velocity $\overline{\omega}$ and its origin O' moves relative to XYZ. If $\overline{r}$ is the position vector of P and $\overline{R}$ is the position vector of O', both relative to O, then

$$\overline{r} = \overline{R} + \overline{\rho} \tag{2-50}$$

where $\overline{\rho}$ is the position vector of P relative to O'.

Differentiating with respect to time we obtain the absolute velocity.

$$\overline{v} = \dot{\overline{r}} = \dot{\overline{R}} + \dot{\overline{\rho}} \tag{2-51}$$

But the derivative $\dot{\overline{\rho}}$ can be expressed in terms of its value relative to xyz by using Equation (2-46), resulting in

$$\dot{\overline{\rho}} = (\dot{\overline{\rho}})_r + \overline{\omega} \times \overline{\rho} \tag{2-52}$$

Then, from Equations (2-51) and (2-52) we obtain that

$$\overline{v} = \dot{\overline{R}} + (\dot{\overline{\rho}})_r + \overline{\omega} \times \overline{\rho} \tag{2-53}$$

To obtain the absolute acceleration of P, we find the rate of change of each of the terms in Equation (2-53), as viewed by an observer

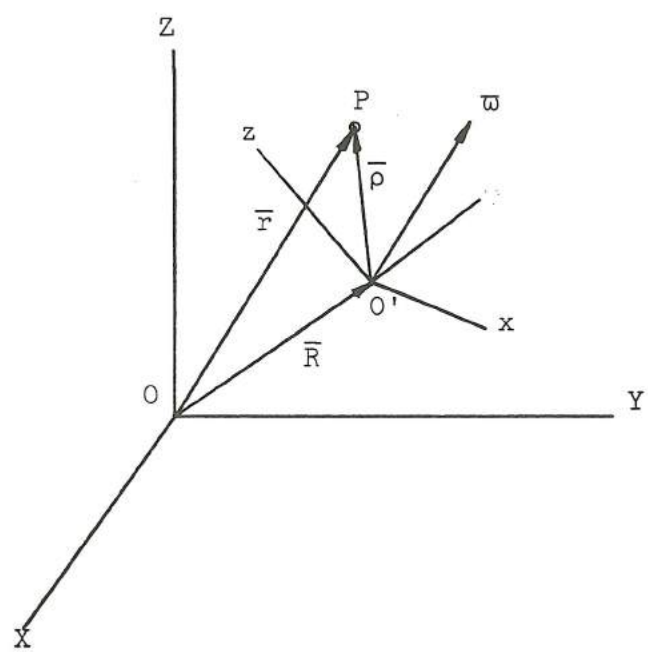

Figure 2-9

fixed in system XYZ. Equation (2-46) is used in the differentiation of the last two terms where vectors relative to the rotating system are desired.

Thus we obtain

$$\frac{d}{dt}(\dot{\overline{R}}) = \ddot{\overline{R}} \qquad (2\text{-}54)$$

$$\frac{d}{dt}[(\dot{\overline{\rho}})_r] = (\ddot{\overline{\rho}})_r + \omega \times (\dot{\overline{\rho}})_r \qquad (2\text{-}55)$$

$$\frac{d}{dt}(\overline{\omega} \times \overline{\rho}) = \dot{\overline{\omega}} \times \overline{\rho} + \overline{\omega} \times (\dot{\overline{\rho}})_r + \overline{\omega} \times (\overline{\omega} \times \overline{\rho}) \qquad (2\text{-}56)$$

Adding equations (2-54) through (2-56), we find the absolute acceleration of P.

$$\overline{a} = \ddot{\overline{R}} + (\ddot{\overline{\rho}})_r + \dot{\overline{\omega}} \times \overline{\rho} + \overline{\omega} \times (\overline{\omega} \times \overline{\rho}) + 2\overline{\omega} \times (\dot{\overline{\rho}})_r \qquad (2\text{-}57)$$

Note that $\dot{\overline{\omega}}$, as well as $\overline{\omega}$, is the same when viewed from either coordinate system, i.e., $\overline{\omega} = (\overline{\omega})_r$ and also $\dot{\overline{\omega}} = (\dot{\overline{\omega}})_r$, the latter because $\overline{\omega} \times \overline{\omega} = 0$.

The nature of each of the terms comprising the total acceleration is as follows:

$\ddot{\overline{R}}$ is the absolute acceleration of the origin O' of the moving system.

$(\ddot{\overline{\rho}})_r$ is the acceleration of P as viewed by an observer on the moving system.

$\dot{\overline{\omega}} \times \overline{\rho}$ is the tangential acceleration of P (more properly, the rate of change of tangential velocity) due to a changing magnitude or direction of $\overline{\omega}$.

$\overline{\omega} \times (\overline{\omega} \times \overline{\rho})$ is the centripetal acceleration of P due to $\overline{\omega}$, considering P fixed at its instantaneous position relative to xyz and ignoring any centripetal components of $\ddot{\overline{R}}$.

$2\,\overline{\omega} \times (\dot{\overline{\rho}})_r$ is the Coriolis acceleration of P. It comes from two sources. The term in Equation (2-55) is due to the changing direction in space of the velocity relative to the moving system. The term in Equation (2-56) is the rate of change of the tangential velocity due to changing magnitude or direction of the position vector $\overline{\rho}$ relative to the moving system.

2-7. Examples

Example 2-1. A rigid tube of length 2ℓ rotates at a constant rate about a transverse axis through its center. A particle P moves at a constant speed v relative to the tube.

a. Solve for the absolute acceleration of P in terms of the unit vectors $\overline{e}_r$ and $\overline{e}_\phi$.

b. Suppose that water of density ρ is introduced through the axis of rotation and flows radially out each end with relative speed v. For a tube of unit internal cross-section, find the external moment necessary to rotate the tube at constant $\overline{\omega}$.

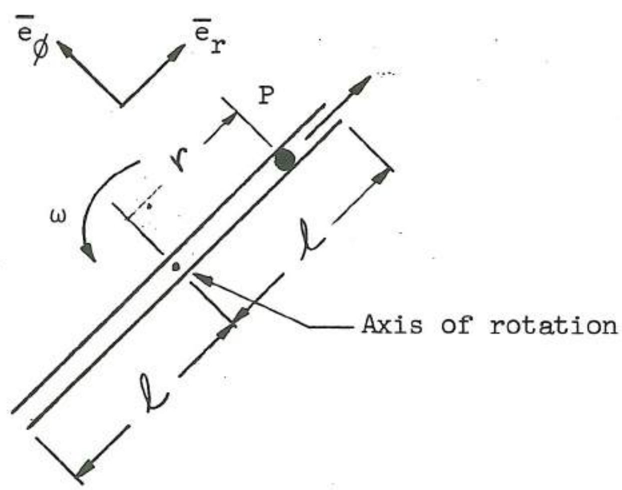

Figure 2-10

The solution to part (a) can be obtained directly from Equation (2-21) giving the acceleration in a cylindrical coordinate system. Noting that

$$\dot{r} = v = \text{constant}$$

and

$$\dot{\phi} = \omega = \text{constant}$$

we obtain

$$\overline{a} = -r\omega^2\overline{e}_r + 2\,v\,\omega\,\overline{e}_\phi \qquad (r > 0)$$

the first term being the centripetal acceleration and the second term being the Coriolis acceleration of P.

For part (b), consider a mass element $\rho\,dr$ at P. We saw in part (a) that it has a tangential acceleration

$$a_\phi = 2\,v\omega$$

and, therefore, a tangential force component

$$dF_{\phi} = 2 \rho v \omega dr$$

must be applied to it in accordance with Newton's laws. Taking the moment of this force about the axis of rotation and integrating, we find that the total applied moment (counter-clockwise) is

$$M = 2 \int_0^{\ell} 2 \rho v \omega r dr = 2 \rho \ell^2 v\omega$$

where the integral is taken over half the tube and then doubled. We have omitted consideration of the radial component of acceleration because it is associated with radial forces which have no moment about the axis of rotation.

Example 2-2. Suppose that a particle P moves along a line of longitude on a sphere of radius a rotating at constant $\overline{\omega}$. If its speed relative to the sphere is

$$v = kt$$

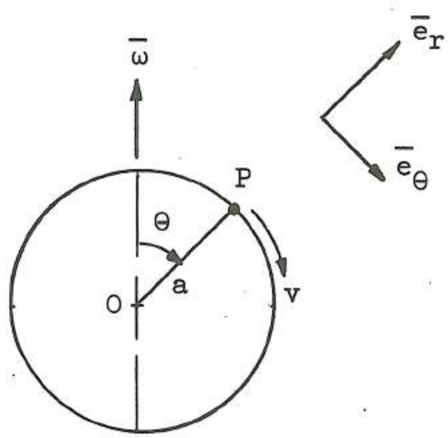

Figure 2-11

and if the center of the sphere is fixed, find the absolute acceleration of P in terms of spherical unit vectors

$$\overline{e}_r, \overline{e}_{\theta}, \overline{e}_{\phi}. \text{ Let } \theta(0) = \theta_o.$$

In solving this problem first note that

$$\dot{\theta} = \frac{v}{a} = \frac{kt}{a}$$

from which we obtain that

$$\theta = \theta_o + \frac{kt^2}{2a}$$

Also

$$r = a$$

$$\dot{\phi} = \omega$$

The angle ϕ does not enter the problem because the unit vectors rotate with the sphere.

Performing the necessary differentiations and substituting into the general acceleration equation in spherical coordinates, Equation (2-29), we obtain the acceleration components

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\,\dot{\phi}^2 \sin^2\theta = -\frac{k^2t^2}{a} - a\,\omega^2 \sin^2\theta$$

$$a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} - r\,\dot{\phi}^2 \sin\theta\cos\theta = k - a\,\omega^2 \sin\theta\cos\theta$$

$$a_\phi = r\,\ddot{\phi}\sin\theta + 2\,\dot{r}\,\dot{\phi}\sin\theta + 2\,r\,\dot{\theta}\,\dot{\phi}\cos\theta = 2\,k\,\omega\,t\cos\theta$$

giving the acceleration

$$\bar{a} = \left(-\frac{k^2t^2}{a} - a\,\omega^2\sin^2\theta\right)\bar{e}_r + (k - a\,\omega^2\sin\theta\cos\theta)\bar{e}_\theta$$
$$+ (2\,k\omega t\cos\theta)\bar{e}_\phi$$

Now let us solve this example using the general vector relation for a moving coordinate system as given in Equation (2-57). Choose the moving coordinate system to have its origin at 0 and to rotate with the sphere at angular velocity $\bar{\omega}$. Evaluating each of the terms we find that

$$\ddot{\bar{R}} = 0$$

$$(\ddot{\bar{\rho}})_r = -\frac{k^2t^2}{a}\,\bar{e} + k\,\bar{e}_\theta$$

$$\dot{\bar{\omega}} \times \bar{\rho} = 0$$

$$\bar{\omega} \times (\bar{\omega} \times \bar{\rho}) = a\,\omega^2\sin\theta\,(-\sin\theta\,\bar{e}_r - \cos\theta\,\bar{e}_\theta)$$

$$2\,\bar{\omega} \times (\dot{\bar{\rho}})_r = 2\,k\,\omega\,t\cos\theta\,\bar{e}_\phi$$

Adding these terms we again obtain the result given above.

Example 2-3. Find the acceleration of point P on the circumference of a wheel of radius r_2 rolling on the inside of a fixed circular cylinder of radius r_1. An arm connecting the fixed point O and the wheel hub at O' moves at a constant angular velocity ω. The position of P relative to the arm is given by the angle ϕ.

The solution will be found with the aid of Equation (2-57), assuming that the moving system is fixed in moving arm with its origin at the wheel hub O' The unit vectors $\overline{e}_t$ and $\overline{e}_n$ rotate with the arm.

Before proceeding further, let us find the relation between ϕ and ω. As an aid, let us introduce the notion of the instantaneous center of rotation. We saw earlier in Section 2-1 that the instantaneous velocities of all points in a rigid body are known if, at a given time, the velocity of the base point and also the angular velocity of the body are known. In case the body motion is such that all points move in parallel planes, then the body is said to have plane motion. In this case the angular velocity $\overline{\omega}$ has a fixed direction in space that is perpendicular to the parallel planes of particle motion. Also, at any given time, it is possible to find an instantaneous axis of rotation that is fixed in space and about which the rigid body appears to be rotating with angular velocity ω. For an essentially plane body moving in its own plane, this reduces to rotation about a stationary base point known as the instantaneous center of rotation.

In the example being considered here, we have a particular type of plane motion known as rolling motion, in which there is no relative motion between two bodies at their point of contact, i.e., there is no slipping. Since one of the bodies (the cylinder) is stationary, the point of contact C on the wheel must also be stationary and is thus the instantaneous center of rotation.

The absolute velocity of any point on the wheel may be calculated by considering the motion to result from rotation at its absolute angular rate about its instantaneous center C. So the velocity of the hub O' is

$$v_{O'} = r_2(\dot{\phi} - \omega)\overline{e}_t$$

since the absolute angular velocity of the wheel is the sum of its angular velocity relative to the arm and the absolute angular velocity of the arm.

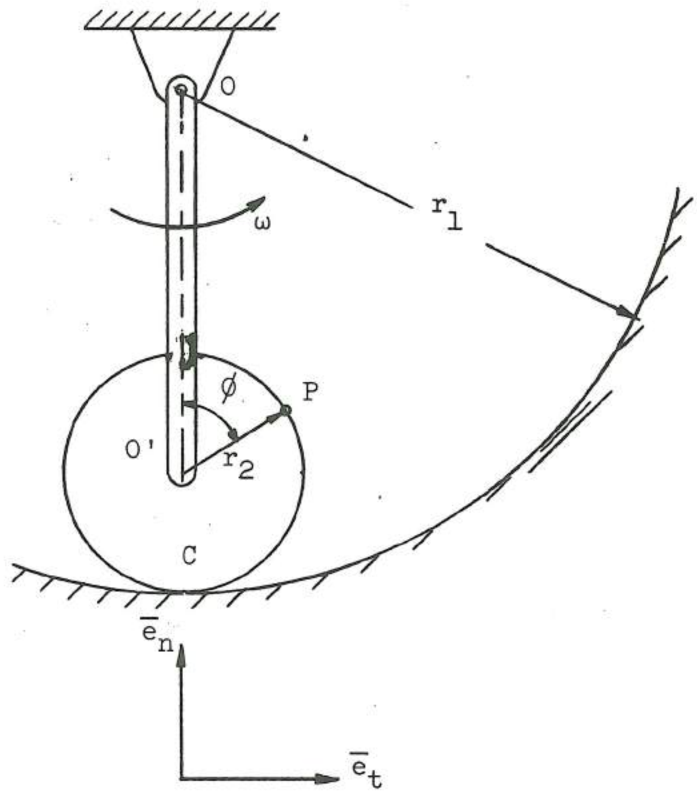

Figure 2-12

But the velocity of O' can also be calculated by considering the arm alone.

$$\bar{v}_{O'} = (r_1 - r_2)\ \omega\ \bar{e}_t$$

From these two equations we obtain that

$$\dot{\phi} = \frac{r_1}{r_2}\ \omega$$

Now let us proceed with the evaluation of the absolute acceleration of P, using Equation (2-57). We find that

$$\ddot{\bar{R}} = (r_1 - r_2)\ \omega^2 \bar{e}_n$$

which is just the centripetal acceleration due to the uniform circular motion of O' about O. Next, we evaluate the acceleration of P relative to the moving system. It is the centripetal acceleration due to the uniform circular motion of P at the angular rate $\dot{\phi}$.

$$(\ddot{\bar{\rho}})_r = r_2\ \dot{\phi}^2\ (-\cos\phi\ \bar{e}_n - \sin\phi\ \bar{e}_t)$$
$$= \frac{r_1^2\omega^2}{r_2}(-\cos\phi\ \bar{e}_n - \sin\phi\ \bar{e}_t)$$

The term $\dot{\bar{\omega}} \times \bar{\rho}$ is zero since $\bar{\omega}$ is constant. The acceleration of P due to the rotation of the moving system about O' is the centripetal acceleration

$$\bar{\omega} \times (\bar{\omega} \times \bar{\rho}) = r_2\ \omega^2\ (-\cos\phi\bar{e}_n - \sin\phi\ \bar{e}_t)$$

Finally, the Coriolis acceleration is

$$2\ \bar{\omega} \times (\dot{\bar{\rho}})_r = 2\ r_2\ \omega\ \dot{\phi}(\cos\phi\ \bar{e}_n + \sin\phi\ \bar{e}_t)$$

since the velocity of P as seen from the moving system is

$$(\dot{\bar{\rho}})_r = r_2\ \dot{\phi}\ (\cos\phi\ \bar{e}_t - \sin\phi\bar{e}_n)$$

Adding the individual acceleration terms, we find the absolute acceleration of P.

$$\bar{a} = [(r_1 - r_2)\omega^2 + \cos\phi\ (2\ r_1 - r_2 - \frac{r_1^2}{r_2})\omega^2]\bar{e}_n$$
$$+ [\sin\phi\ (2\ r_1 - r_2 - \frac{r_1^2}{r_2})\omega^2]\ \bar{e}_t$$

A few comments should be made concerning the choice of the moving coordinate system in the above examples. Although Equation (2-57) is valid for an arbitrary motion of the moving coordinate system, it should be chosen such that the calculations are made as simple as possible. Roughly speaking, the motion of point P relative to the moving system should be of about the same complexity as the absolute motion of O', providing that the angular velocity $\overline{\omega}$ is constant or varies in a simple fashion. Also, the choice of unit vectors in expressing the result should be chosen for convenience. With rare exceptions, they should form an orthogonal set.

When one uses the concept of instantaneous center of rotation in forming kinematic relationships, much care should be taken if quantities other than velocities are being calculated. In particular, it should be pointed out that even though the instantaneous center is stationary, its acceleration may not be zero. For example, when a wheel rolls along a straight line, the point on the wheel at the instantaneous center of rotation has a centripetal acceleration of magnitude $r\omega^2$ even when its velocity is zero. Its path in space is a cycloid and the point becomes the instantaneous center of rotation when it is reversing its direction at the cusp of the cycloid.

CHAPTER 3

DYNAMICS OF A PARTICLE

In Chapter 2 we used the methods of kinematics to obtain the absolute acceleration of a particle and, assuming a knowledge of its mass, we were able to calculate the total external force acting on the particle by using Equation (1-28). In this chapter we will consider the reverse problem, namely, the problem of calculating the motion of a particle from a knowledge of the external forces acting upon it.

First, let us consider the general case where the force acting on a particle is a function of its position and velocity and the time. From Equation (1-28), its differential equation of motion is

$$m\ddot{\bar{r}} = \bar{F}\ (\bar{r},\dot{\bar{r}},t) \qquad (3\text{-}1)$$

Knowing the function $\bar{F}\ (\bar{r},\dot{\bar{r}},t)$ and the mass m, we would like to solve for the position $\bar{r}$ as an explicit function of time. Unfcrtunately, an analytic solution of this equation is impossible except in special cases.

To see the difficulties more clearly, let us write the vector equation in terms of its Cartesian components.

$$\begin{aligned} m\ddot{x} &= F_x\ (x,y,z,\dot{x},\dot{y},\dot{z},t) \\ m\ddot{y} &= F_y\ (x,y,z,\dot{x},\dot{y},\dot{z},t) \\ m\ddot{z} &= F_z\ (x,y,z,\dot{x},\dot{y},\dot{z},t) \end{aligned} \qquad (3\text{-}2)$$

The force components F_x, F_y, and F_z are, in general, nonlinear functions of the coordinates, velocities, and time, and thus the equations are hopelessly complex from the standpoint of obtaining an analytical solution.

Nevertheless, it is the thesis of Newtonian mechanics that a complete knowledge of the forces on a particle determines its future motion, providing that the initial values of displacement and velocity are known. (Using Cartesian coordinates, this would imply six initial conditions.) So a solution to the problem does, in fact, exist. With the aid of modern electronic computers, and using approximate methods, it is possible to obtain solutions to the complete equations that are of sufficient accuracy for engineering purposes.

Any general analytical solution of equations of the form of Equation (3-2) will contain six arbitrary coefficients which are evaluated from the six initial conditions. One method of obtaining the general solution is to look for integrals or constants of the motion, that is, attempt to find six functions of the form

$$f_k(x,y,z,\dot{x},\dot{y},\dot{z},t) = \alpha_k \quad (k = 1, 2, \ldots, 6) \tag{3-3}$$

where the α_k are all constant. If the functions are all distinct, i.e., none are derivable from the others, then in principle they may be solved for the displacement and velocity of the particle as a function of time and the constants α_k.

It is normally not possible to obtain all the α_k by any direct process. However, one of the principal topics of advanced classical mechanics is the finding of proper coordinate transformations such that the solution for the constants in terms of the new coordinates is a straightforward process.

Sometimes the constants of the motion can be given a simple physical interpretation, thereby giving us more insight into the nature of the motion. For example, a constant might be the total energy or the angular momentum about a given point. Even in cases where we do not completely solve for the motion, a knowledge of some of the constants that are applicable to the given problem may help us to obtain results such as limiting values of certain coordinates.

In this chapter we will discuss some of the simpler methods and principles to be used in solving for the motion of a particle. As will be seen in the following chapters, these principles can be expanded to apply to systems of particles and to rigid body motion, and will thus form an important part of our treatment of the subject of mechanics.

3-1. Direct Integration of the Equations of Motion

Returning now to the general equation of motion as given by Equation (3-1), we will consider several cases in which a direct integration can be used to find the motion of the particle.

<u>Case 1: Constant Acceleration</u>. The simplest case is that in which the external force on the particle is constant in magnitude and direction. If we consider the Cartesian components of the motion, we find from Equation (3-2) that

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z \end{aligned} \tag{3-4}$$

where F_x, F_y, and F_z are each constant.

We see from Equation (3-4) that the motions in the x, y, and z directions are independent, so now let us consider just the motion parallel to the x-axis. Denoting the x components of acceleration and velocity by a and v, respectively, we find that

$$a = \frac{F_x}{m} \tag{3-5}$$

and, by direct integration with respect to time, we obtain that

$$v = v_o + at \tag{3-6}$$

and

$$x = x_o + v_o t + \frac{1}{2} at^2 \tag{3-7}$$

where x_o and v_o are the displacement and velocity components in the x direction at t = 0. Of course, similar equations would apply to the motion in the direction of the y and z axes.

The time required for the particle to attain a given speed v or displacement x is found by solving Equations (3-6) and (3-7) for t.

$$t = \frac{1}{a} (v - v_o) \tag{3-8}$$

$$t = \frac{1}{a} \left[\sqrt{2a(x-x_o) + v_o^2} - v_o \right] \tag{3-9}$$

An equation relating speed and displacement for this case of constant acceleration can be obtained by eliminating the time t between Equations (3-8) and (3-9), giving the result

$$v^2 = v_o^2 + 2a(x - x_o) \tag{3-10}$$

<u>Case 2: $\overline{F} = \overline{F}(t)$</u>. Suppose now that the external force is a function of time only. Again, the general equation can be written in terms of three independent equations giving orthogonal components of the motion. For example, if we consider only the x component, the equation of motion becomes,

$$m\ddot{x} = F_x(t) \tag{3-11}$$

which can be integrated directly to give the displacement

$$x = x_o + v_o t + \frac{1}{m} \int_0^t \left[\int_0^{t_2} F_x (t_1) dt_1 \right] dt_2 \tag{3-12}$$

where again v_o is the initial velocity component in the x direction.

Case 3: $\overline{F} = F_x(x)\overline{i} + F_y(y)\overline{j} + F_z(z)\overline{k}$. For this case we will again consider only the x component of the motion, noting that similar results apply to the motion in the y and z directions. The differential equation for the motion is

$$m\ddot{x} = F_x(x) \tag{3-13}$$

Making the substitution

$$\dot{x} = v \tag{3-14}$$

we find that

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{3-15}$$

From Equations (3-13) and (3-15) we find that

$$mv\frac{dv}{dx} = F_x(x) \tag{3-16}$$

which can be integrated directly to give

$$\frac{m}{2}(v^2 - v_o^2) = \int_{x_o}^{x} F_x(x)dx \tag{3-17}$$

This result is a special application of the general principle of work and kinetic energy, as we will see in Section 3-2.

Equation (3-17) can be integrated again to give a solution of the form

$$f(x_o, v_o, x) = t \tag{3-18}$$

from which we can obtain

$$x = g(x_o, v_o, t) \tag{3-19}$$

Suppose, for example, that the force is derived from the extension of a linear spring, i.e.,

$$F_x = -kx \tag{3-20}$$

where k is a constant. Then

$$\frac{m}{2}(v^2 - v_o^2) = -\int_{x_o}^{x} kx\,dx = -\frac{1}{2}k(x^2 - x_o^2) \tag{3-21}$$

From Equations (3-14) and (3-21) we can write

$$\frac{dx}{dt} = \sqrt{v_o^2 - \frac{k}{m}\,(x^2 - x_o^2)}$$

or

$$\sqrt{\frac{m}{k}} \int_{x_o}^{x} \frac{dx}{\sqrt{\frac{m\,v_o^2}{k} + x_o^2 - x^2}} = t$$

This can be integrated to give

$$\sqrt{\frac{m}{k}} \left[\sin^{-1}\left(x/\sqrt{\frac{m}{k}\,v_o^2 + x_o^2}\right) - \sin^{-1}\left(x_o/\sqrt{\frac{m}{k}\,v_o^2 + x_o^2}\right) \right] = t$$

or

$$x = \sqrt{\frac{m}{k}\,v_o^2 + x_o^2}\;\sin\left(\sqrt{\frac{k}{m}}\,t + \alpha\right) \tag{3-22}$$

where

$$\alpha = \sin^{-1}\left(x_o/\sqrt{\frac{m}{k}\,v_o^2 + x_o^2}\right) \tag{3-23}$$

This is the solution for harmonic motion in one dimension with arbitrary initial conditions.

Case 4. $\overline{F} = F_x(\dot{x})\overline{i} + F_y(\dot{y})\overline{j} + F_z(\dot{z})\overline{k}$. In the analysis of this case we will again consider just the x component of the motion and note that similar results apply to the other components. The equation of motion is

$$m\,\frac{dv}{dt} = F_x(v) \tag{3-24}$$

from which we obtain

$$m \int_{v_o}^{v} \frac{dv}{F_x(v)} = t \tag{3-25}$$

Equation (3-24) can also be written in the form

$$mv\,\frac{dv}{dx} = F_x(v) \tag{3-26}$$

from which we obtain

$$m \int_{v_o}^{v} \frac{v \ dv}{F_x(v)} = x - x_o \tag{3-27}$$

By eliminating v between Equations (3-25) and (3-27) we can solve for x in the form

$$x = f(x_o, v_o, t) \tag{3-28}$$

This result can also be obtained by solving Equation (3-25) for v as a function of t and then integrating again to obtain x as a function of t.

As an example of Equation (3-24), let us consider the one-dimensional motion of a particle moving against a linear damping force, i.e.,

$$F_x(v) = -c \ v \tag{3-29}$$

where c is a constant. In this case

$$- \frac{m}{c} \int_{v_o}^{v} \frac{dv}{v} = t$$

or

$$\ln(\frac{v}{v_o}) = - \frac{c}{m} t$$

which can also be written

$$v = v_o \ e^{-\frac{ct}{m}} \tag{3-30}$$

Integrating again with respect to time we obtain

$$x = x_o + \frac{m}{c} v_o (1 - e^{-\frac{ct}{m}}) \tag{3-31}$$

From Equation (3-27) we can obtain the expression relating speed and displacement, namely,

$$\frac{m}{c} (v_o - v) = x - x_o \tag{3-32}$$

which can be verified from Equations (3-30) and (3-31).

3-2. Work and Kinetic Energy

We turn now to the presentation of some of the general principles of particle mechanics. For the cases where these principles apply, they are directly derivable from Newton's laws of motion and thus contain no new information. On the other hand they promote further insight into the nature of particle motion.

The first of these principles to be presented is the principle that the increase in the kinetic energy of a particle in going from one point to another is equal to the work done on the particle by the external forces acting over the same interval.

Suppose, for example, that a particle of mass m moves from A to B under the action of an arbitrary force $\overline{F}$, as shown in Figure 3-1. Starting with Newton's law

$$\overline{F} = m\,\ddot{\overline{r}}$$

we form the line integral over the path from A to B.

$$\int_A^B \overline{F}\cdot d\overline{r} = \int_A^B m\,\ddot{\overline{r}}\cdot d\overline{r} \tag{3-33}$$

But

$$\frac{d}{dt}\left(\dot{\overline{r}}\cdot\dot{\overline{r}}\right) = 2\ddot{\overline{r}}\cdot\frac{d\overline{r}}{dt}$$

So

$$\int_A^B m\,\ddot{\overline{r}}\cdot d\overline{r} = \frac{m}{2}\int_A^B d(v^2) = \frac{m}{2}\left(v_B^2 - v_A^2\right) \tag{3-34}$$

From Equations (3-33) and (3-34) we obtain the result

$$\int_A^B \overline{F}\cdot d\overline{r} = \frac{1}{2}\,m\,v_B^2 - \frac{1}{2}\,m\,v_A^2 \tag{3-35}$$

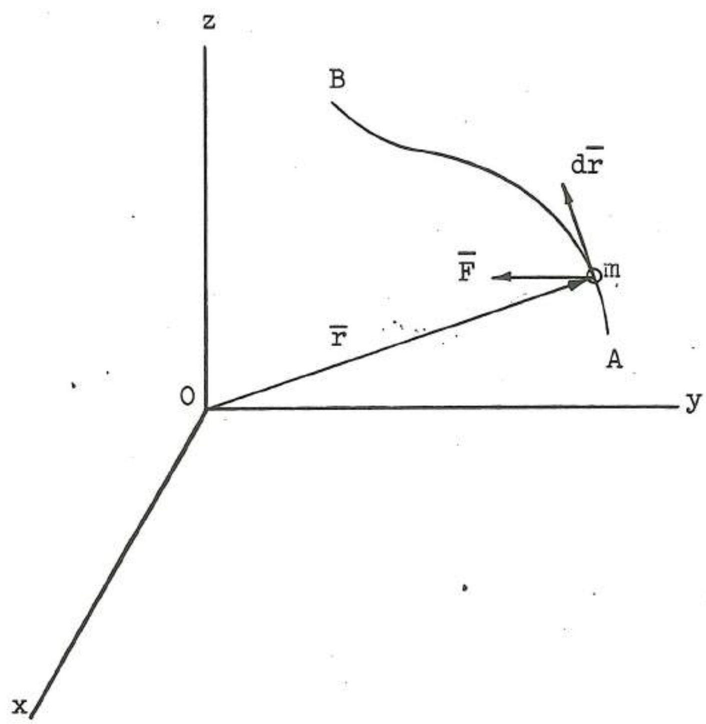

Figure 3-1

The integral on the left side of this equation is the work done on the particle by the resultant of the external forces as the particle moves over the path followed from A to B.

$$W = \int_A^B \overline{F}\cdot d\overline{r} \tag{3-36}$$

In general, the kinetic energy of a particle relative to an inertial system is

$$T = \frac{1}{2} m v^2 \tag{3-37}$$

where v is the speed of the particle relative to that system. Therefore, the right side of Equation (3-35) represents the increase in kinetic energy in going from A to B.

Using Equations (3-36) and (3-37) we could write Equation (3-35) in the form

$$W = T_B - T_A \tag{3-38}$$

It should be emphasized that calculations of work, kinetic energy, and also changes in kinetic energy are dependent upon which inertial system is used as a frame of reference. On the other hand, measurements of force and also of time (in Newtonian mechanics) are the same for all observers.

Example 3-1. Suppose that a whirling particle of mass m is pulled by a string toward a fixed center at O in such a manner that the radial component of velocity is small compared to the tangential component. Also, $\ddot{r}$ can be neglected relative to the centripetal acceleration.

Find the change in the angular rate ω as r decreases.

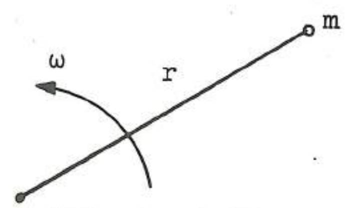

Figure 3-2

We will use the principle of work and kinetic energy to solve this problem. With the above assumptions the acceleration of the particle is entirely centripetal and the total external force is the radial force

$$F_r = -m\, r\, \omega^2$$

The work done on the particle in an infinitesimal displacement dr is

$$dW = F_r\, dr = -m\, \omega^2\, r\, dr$$

This must be equal to the change in kinetic energy

$$dT = m\, v\, dv = m\, r\, \omega(\omega\, dr + r\, d\omega)$$

found by differentiating Equation (3-37).

The principle of work and kinetic energy applies to infinitesimal changes and states that

$$dW = dT$$

or

$$-m\omega^2\, r\, dr = m\omega^2\, r\, dr + mr^2\omega\, d\omega$$

which can be rearranged and integrated in the interval r_o to r, corresponding to the angular rates ω_o and ω.

$$-2\int_{r_o}^{r} \frac{dr}{r} = \int_{\omega_o}^{\omega} \frac{d\omega}{\omega}$$

or

$$-2\ \ln\left(\frac{r}{r_o}\right) = \ln\left(\frac{\omega}{\omega_o}\right)$$

from which we obtain

$$\omega = \left(\frac{r_o}{r}\right)^2 \omega_o$$

We could also have written the final result in the form

$$mr^2\omega = mr_o^2\omega_o$$

which is in agreement with the conservation of angular momentum principle to be developed in Section 3-6.

3-3. Conservation of Mechanical Energy.

Referring again to Figure 3-1, let us suppose that the force $\overline{F}$ acting on the particle has the following characteristics:

(1) it is a function of position only, and

(2) the line integral $\int_A^B \overline{F}\cdot d\overline{r}$ is independent of the path taken between A and B.

This last statement also implies that

$$\oint \overline{F}\cdot d\overline{r} = 0 \qquad (3\text{-}39)$$

where the integral is taken around any closed path. In other words, the work done by the force F in going around an arbitrary closed path is zero. Furthermore,

$$\int_A^B \overline{F}\cdot d\overline{r} = -\int_B^A \overline{F}\cdot d\overline{r} \qquad (3\text{-}40)$$

so that reversing the direction of travel along a given path merely changes the sign of the work done.

A force with the above characteristics is said to be a conservative force, i.e., it forms a conservative force field. Practically speaking, this means that the force is not dissipative, and that any mechanical process taking place under its influence is reversible.

So, if W, as given by Equation (3-36), is found to depend only upon the location of the end points, then the integrand must be an exact differential.

$$\overline{F}\cdot d\overline{r} = -dV \qquad (3\text{-}41)$$

where the minus sign has been chosen for convenience in the statement of later results.

Therefore, we find that

$$W = \int_A^B \overline{F}\cdot d\overline{r} = -\int_A^B dV = V_A - V_B \qquad (3\text{-}42)$$

which may be combined with Equation (3-38) to give

$$T_A + V_A = T_B + V_B = E \qquad (3\text{-}43)$$

where E is the total energy. The scalar V is a function of position only for a given particle and is known as the potential energy of the particle. The difference in potential energy values between any two points indicates the amount of work done by the conservative force field on the particle as it proceeds from one point to the other, as can be seen from Equation (3-42). The total energy E is constant for a conservative system and is the sum of the kinetic and potential energies.

Equation (3-43) is a mathematical statement of the principle of conservation of mechanical energy and applies to systems in which the only forces that do work on the particle are those arising from a conservative force field. Workless forces such as those arising from frictionless, fixed constraints do not change the applicability of the principle.

3-4. Potential Energy

Now let us take a closer look at the potential energy function V. We saw that V is a function of position only. So, if we express the position of a particle in terms of its Cartesian coordinates, we find that

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}dz \qquad (3\text{-}44)$$

Also we note that

$$\overline{F} \cdot d\overline{r} = F_x\,dx + F_y\,dy + F_z\,dz \qquad (3\text{-}45)$$

From Equations (3-41), (3-44), and (3-45), we find that

$$\begin{aligned} F_x &= -\frac{\partial V}{\partial x} \\ F_y &= -\frac{\partial V}{\partial y} \\ F_z &= -\frac{\partial V}{\partial z} \end{aligned} \qquad (3\text{-}46)$$

since the equality holds for an arbitrary infinitesimal displacement. Therefore,

$$\overline{F} = F_x\,\overline{i} + F_y\,\overline{j} + F_z\,\overline{k} = -\frac{\partial V}{\partial x}\overline{i} - \frac{\partial V}{\partial y}\overline{j} - \frac{\partial V}{\partial z}\overline{k} \qquad (3\text{-}47)$$

But the gradient of the scalar function V can be written as

$$V = \frac{\partial V}{\partial x}\,\overline{i} + \frac{\partial V}{\partial y}\,\overline{j} + \frac{\partial V}{\partial k}\,\overline{k} \qquad (3\text{-}48)$$

So we find that the force at any point due to the conservative force field is

$$\overline{F} = -\nabla V \qquad (3\text{-}49)$$

This means that the force is in the direction of the largest spatial rate of decrease of V and is equal in magnitude to that rate of decrease.

Equation (3-49) is a general vector equation, so the gradient need not be expressed in terms of Cartesian coordinates. For a general orthogonal coordinate system, the component of the force $\overline{F}$ in the direction of the unit vector $\overline{e}_1$ is given by

$$F_1 = -\frac{\partial V}{\partial x_1} \qquad (3\text{-}50)$$

where x_1 is the linear displacement in the direction $\overline{e}_1$ at the point under consideration.

Inverse-square attraction. As an example of a potential energy calculation, consider the case of the inverse-square attraction of a particle of mass m toward a fixed point. The force exerted by the attracting field is entirely radial and is equal to

$$F_r = -\frac{\partial V}{\partial r} = -\frac{K}{r^2} \qquad (3\text{-}51)$$

where r is the distance of the particle from the attracting center. Since the force is a function of r only, we can integrate Equation (3-51) directly to give

$$V = -\frac{K}{r} + C \qquad (3\text{-}52)$$

where C is the arbitrary constant of integration. In the usual case we choose $C = 0$, implying that the potential energy is always negative and approaches zero as r approaches infinity.

The fact that the potential energy contains an arbitrary constant would seem to require that all measurable quantities of the motion such as velocity, acceleration, etc., should be independent of the choice of C since the motion in a given situation is not arbitrary. That this is actually true is confirmed by the fact that potential energy enters all computations as a potential energy difference, in which case the constant C cancels out. The mode of entry into calculations is always essentially as work, as in Equation (3-42).

As a result, the choice of C, i.e., the choice of the datum or reference point of zero potential energy, is made for convenience in solving the problem at hand.

Gravitational potential. The most commonly encountered inverse-square force in the study of mechanics is the force of gravitational attraction. As we have seen, the gravitational potential energy must be of the form given by Equation (3-52).

Now let us consider the particular case of gravitational attraction by the earth. Assuming the earth to be a sphere whose density is a function only of the radial distance from its center, it can be shown that the attractive force on an external particle is the same as if the entire mass of the earth were concentrated at its center.

So, from Equation (3-51) the gravitational force on a particle of mass m outside the earth's surface is the radial force

$$F_r = -\frac{K}{r^2} \quad (r \geq R) \tag{3-53}$$

where R is the radius of the earth (Figure 3-3). The constant K can be evaluated from the knowledge that the weight w of the particle is the gravitational attraction of the earth on the particle at the earth's surface.

$$-F_r = w = \frac{K}{R^2} \tag{3-54}$$

But the weight is also given by

$$w = mg \tag{3-55}$$

where g is the acceleration of gravity at the surface of a nonrotating earch.

From Equations (3-54) and (3-55) we obtain that

$$K = mg\, R^2 \tag{3-56}$$

Therefore,

$$F_r = -mg\,\frac{R^2}{r^2} \qquad (r \geq R) \tag{3-57}$$

and

$$V = -mg\,\frac{R^2}{r} \qquad (r \geq R) \tag{3-58}$$

We could also write the potential energy in terms of the height h above the earth's surface. Letting

$$r = R + h \tag{3-59}$$

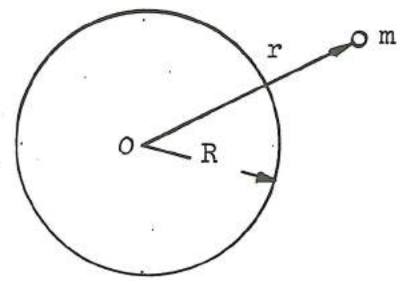

Figure 3-3

we find from Equation (3-58) that

$$V = - mg \frac{R}{(1 + \frac{h}{R})} \tag{3-60}$$

For motion near the surface of the earth, i.e., for $h << R$, this reduces to

$$V \approx - mg\ R\ (1 - \frac{h}{R}) \tag{3-61}$$

But we can eliminate the constant term -mg R by choosing the zero reference for potential energy at the earth's surface. Then we obtain

$$V \approx mgh \qquad (h << R) \tag{3-62}$$

the equation becoming exact for a uniform gravitational field.

This equation is often used in the calculation of local trajectories near the earth's surface, in which case an "inertial" frame is chosen that is fixed in the earth and the value of g includes earth rotation effects at that point.

<u>Linear spring</u>. Another form of potential energy that is commonly encountered is that due to elastic deformation. As an example of elastic potential energy, consider a particle P that is attached by a linear spring of stiffness k to a fixed point O, as shown in Figure 3-4. If the

Figure 3-4

elongation x of the spring is measured from its unstressed position, the particle will experience a force

$$F_x = - \frac{\partial V}{\partial x} = - kx \tag{3-63}$$

Direct integration with respect to x, choosing the zero reference of potential energy at $x = 0$, results in

$$V = \frac{1}{2} kx^2 \tag{3-64}$$

Note that in the above development the force F_x is the force of the spring on a hypothetical particle P and not the force of opposite sign that is applied to the spring. In other words, the emphasis here is on the potential ability of the spring to do work on its surroundings, and not vice versa. A similar comment would apply equally well to gravitational potential energy.

3-5. Impulse and Momentum

In Section 3-2 we obtained the principle of work and kinetic energy for a particle by integrating both sides of Newton's equation of motion with respect to displacement. This principle was found to be particularly useful for cases where the working forces on the particle are a function of position only.

Similarly, we could integrate both sides of the equation of motion with respect to time.

$$\int_{t_1}^{t_2} \bar{F}\, dt = \int_{t_1}^{t_2} m\, \ddot{\bar{r}}\, dt = m\, \bar{v}_2 - m\, \bar{v}_1 \tag{3-65}$$

where the velocity $\bar{v}_2$ corresponds to time t_2 and $\bar{v}_1$ corresponds to t_1, the time interval being arbitrary. The time integral of the force $\bar{F}$ is known as the impulse of the force over the given interval and is designated by

$$\bar{\mathcal{F}} = \int_{t_1}^{t_2} \bar{F}\, dt \tag{3-66}$$

The <u>linear momentum</u> of the particle is the vector $m\bar{v}$. Therefore, we can state the <u>principle of impulse and linear momentum</u>: The change in the linear momentum of a particle during a given interval is equal to the impulse of the external forces acting over the same interval.

$$\bar{\mathcal{F}} = m\, \bar{v}_2 - m\, \bar{v}_1 \tag{3-67}$$

A comparison of this equation with the equation of work and kinetic energy, Equation (3-38), reveals some interesting qualitative differences. First, Equation (3-67) is a vector equation whereas Equation (3-38) is a scalar equation. Its vector nature is an advantage in some cases because it gives the direction as well as the magnitude of the velocity. On the other hand, it may be more difficult to work with.

Another point of interest is that the total impulse $\vec{\mathcal{F}}$ (and consequently the change in linear momentum) is independent of the inertial frame from which the particle motion is viewed. This is in contrast to calculations of work and kinetic energy which, as we have seen, are dependent upon the specific frame of reference.

Equation (3-67) can also be written in terms of its scalar components. For example, if we choose a Cartesian coordinate system in which to express the motion of the particle we find that

$$\begin{aligned} \mathcal{F}_x &= \int_{t_1}^{t_2} F_x \, dt = m \dot{x}_2 - m \dot{x}_1 \\ \mathcal{F}_y &= \int_{t_1}^{t_2} F_y \, dt = m \dot{y}_2 - m \dot{y}_1 \\ \mathcal{F}_z &= \int_{t_1}^{t_2} F_z \, dt = m \dot{z}_2 - m \dot{z}_1 \end{aligned} \qquad (3\text{-}68)$$

where the subscripts 2 and 1 indicate that evaluations of the given velocity components are to be made at t_2 and t_1, respectively. Equation (3-68) follows directly from the equations of motion as given by Equation (3-4).

As a further example of a scalar or one-dimensional form of the impulse and momentum relationship, consider a particle of mass m moving over a prescribed path that is fixed in inertial space, as shown in Figure 3-5. Taking components of force and acceleration along the path, we find from Equations (1-28) and (2-36) that

$$F_s = m \ddot{s} \qquad (3\text{-}69)$$

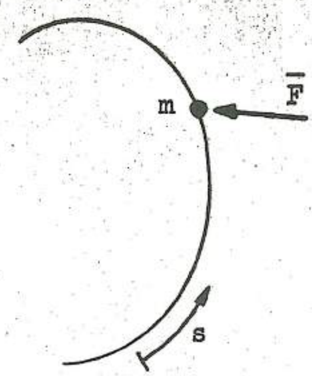

Figure 3-5

where F_s is the component along the instantaneous direction of motion of the total external force $\overline{F}$, and $\ddot{s}$ is the magnitude of the tangential acceleration. Note that frictionless constraint forces are normal to the path and do not contribute to F_s.

Integrating Equation (3-69) with respect to time, we find that

$$\int_{t_1}^{t_2} F_s \; dt = m \, \dot{s}_2 - m \, \dot{s}_1 \qquad (3\text{-}70)$$

where s_2 and s_1 refer to the speed of the particle at times t_2 and t_1, respectively.

Of course, the equation of work and kinetic energy, Equation (3-35), is directly applicable to this system and becomes, in this case,

$$\int_A^B F_s \; ds = \frac{1}{2} m \, \dot{s}_B^2 - \frac{1}{2} m \, \dot{s}_A^2 \qquad (3\text{-}71)$$

3-6. Angular Momentum

We have seen that the linear momentum of a particle with respect to a fixed (inertial) frame is the vector $m\overline{v}$, where m is the mass of the particle and $\overline{v}$ is its absolute velocity. Now let us consider the momentum vector $m\overline{v}$ as a sliding vector whose line of action passes through the particle. If the position vector of the particle with respect to a fixed reference point O is designated by $\overline{r}$, then the moment of momentum or angular momentum about O is given by

$$\overline{H} = \overline{r} \times m\overline{v} \qquad (3\text{-}72)$$

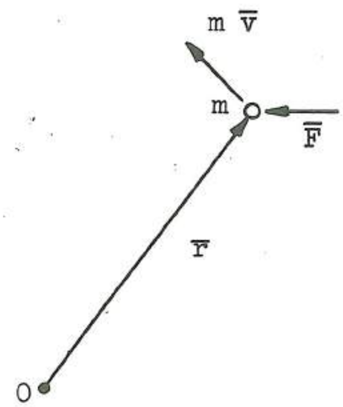

Figure 3-6

Noting that $\overline{v} = \dot{\overline{r}}$, we can differentiate this equation with respect to time, obtaining $\dot{\overline{H}} = \overline{r} \times m\ddot{\overline{r}} + \dot{\overline{r}} \times m\dot{\overline{r}}$ which reduces to

$$\dot{\overline{H}} = \overline{r} \times m\ddot{\overline{r}} \tag{3-73}$$

since the cross product of parallel vectors is zero.

Referring again to the equation of motion

$$\overline{F} = m\ddot{\overline{r}}$$

let us form the cross product of each side with the position vector $\overline{r}$.

$$\overline{r} \times \overline{F} = \overline{r} \times m\ddot{\overline{r}} \tag{3-74}$$

where the left side of Equation (3-74) is identified as the moment $\overline{M}$ about the point O of the total external force $\overline{F}$.

$$\overline{M} = \overline{r} \times \overline{F} \tag{3-75}$$

From Equations (3-73), (3-74), and (3-75) we obtain

$$\dot{\overline{H}} = \overline{M} \tag{3-76}$$

which is a statement of the important principle that the rate of change of angular momentum of a particle about a fixed point is equal to the moment about the same point of the external forces applied to that particle.

For the case where $\overline{M}$ is zero, the angular momentum vector $\overline{H}$ must be constant in magnitude and direction. This is known as the principle of conservation of angular momentum.

The general vector relationship given in Equation (3-76) could also be written in terms of its components. For example, choosing a fixed Cartesian coordinate system, one obtains the scalar equations

$$\dot{H}_x = M_x$$
$$\dot{H}_y = M_y \qquad (3\text{-}77)$$
$$\dot{H}_z = M_z$$

When written in this manner, each equation can be interpreted as relating the moment and rate of change of angular momentum about the corresponding fixed axis passing through the fixed point O.

It is interesting to note that, even though the total angular momentum $\overline{H}$ is not conserved in a given case, one of the components of $\overline{M}$ might be zero, that is, the moment about the corresponding axis might vanish. This would require the angular momentum about that axis to be conserved.

A somewhat similar situation occurs when the motion of a particle is confined to a plane, in which case the angular momentum becomes essentially scalar in nature since its direction is fixed. If the velocity of a particle of mass m (Figure 3-7) has radial and tangential components given by

$$v_r = \dot{r}$$
$$\qquad (3\text{-}78)$$
$$v_\phi = r\omega$$

where ω is the angular velocity of the radius vector as it moves in the plane of particle motion. The angular momentum is of magnitude

$$H = rm\, v_\phi = m\, r^2\, \omega \qquad (3\text{-}79)$$

and, in accordance with the right-hand rule, is directed out of the page. It is independent of v_r since the line of action of the corresponding component of linear momentum passes through the reference point O and thus its moment of momentum is zero.

Now if the mass m is acted upon by radial forces only, regardless of their manner of variation, then the applied moment M will be zero at all times, and therefore, the angular momentum will be conserved.

Since the particle mass is constant, Equation (3-79) can be used to obtain

$$r_1^2 \omega_1 = r_2^2 \omega_2 \tag{3-80}$$

where the subscripts 1 and 2 refer to the values of the variable at arbitrary times t_1 and t_2 respectively.

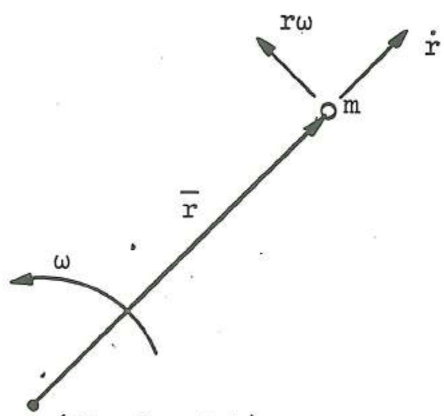

Figure 3-7

Returning now to a general discussion of the angular momentum of a particle, a few more remarks are in order. Since the angular momentum is the moment of the linear momentum vector about a given reference point O, the question arises regarding the proper choice of (1) a reference frame for calculating the linear momentum vector, and (2) a reference point O. Eliminating at the outset, the trivial case where point O is chosen to coincide with the particle, we can state that the reference frame must be inertial, and also that the reference point must be fixed in that frame if Equation (3-76) is to hold for the general motion of a single particle. This follows from the fact that it is based on Newton's equation of motion, and also that Equation (3-73) used in its derivation requires that $\dot{\bar{r}}$ and $\bar{v}$ be parallel.

The question reduces, then, to the choice of a reference point since there is only one inertial frame, at most, in which the point is fixed. Nevertheless, a proper choice from among the possible reference points is very important in clarifying a given problem and simplifying its solution. Often one attempts to choose a reference point such that one or more components of the total angular momentum is conserved.

3-7. Angular Impulse

In a manner similar to that used in obtaining the equation of linear impulse and momentum from Newton's equation of motion, we will now obtain an expression for the change in angular momentum over an arbitrary time interval. Let us integrate both sides of Equation (3-76) with respect to time over the interval t_1 to t_2, obtaining

$$\int_{t_1}^{t_2} \dot{\bar{H}}\, dt = \bar{H}_2 - \bar{H}_1 \qquad (3\text{-}81)$$

and

$$\int_{t_1}^{t_2} \bar{M}\, dt = \bar{\mathcal{M}} \qquad (3\text{-}82)$$

where $\bar{\mathcal{M}}$ is the angular impulse or impulsive moment about a reference point fixed in inertial space. From Equations (3-81) and (3-82) we obtain

$$\bar{\mathcal{M}} = \bar{H}_2 - \bar{H}_1 \qquad (3\text{-}83)$$

which states that the change in the angular momentum of a particle over an arbitrary time interval is equal to the total angular impulse of the external forces acting on the particle during that interval, the reference point being the same fixed point in each computation.

3-8. Examples

Example 3-2. A particle of mass m is suspended vertically by a spring of stiffness k in the presence of a uniform gravitational field, the direction of the gravitational force being as shown by the arrow in Figure 3-8. If the vertical displacement y of the mass is measured from its position when the spring is unstressed, solve for y as a function of time. The mass is released with no velocity at y_o, i.e., $y(0) = y_o$ and $\dot{y}(0) = 0$. Find the maximum values of kinetic energy and potential energy in the ensuing motion.

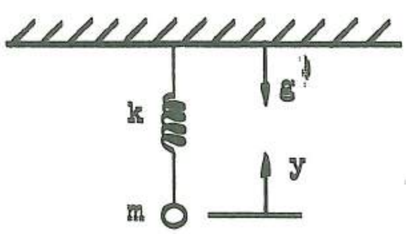

Figure 3-8

The equation of motion can be written with the aid of the free-body diagram shown in Figure 3-9, indicating that the spring force and the gravity force are the only external forces acting on the particle. Using Newton's equation we can write

$$m\ddot{y} = -ky - mg$$

or

$$m\ddot{y} + ky = -mg$$

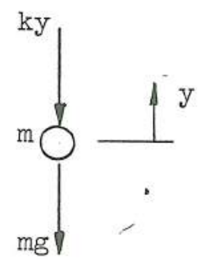

Figure 3-9

The solution to this differential equation is the sum of the transient solution y_t which is the solution of the homogeneous equation

$$m\ddot{y} + ky = 0$$

and the steady state solution y_s (the particular integral) which a solution that fits the complete equation.

In this case

$$y_t = C_1 \cos \sqrt{\frac{k}{m}}\, t + C_2 \sin \sqrt{\frac{k}{m}}\, t$$

and

$$y_s = -\frac{mg}{k}$$

So

$$y = y_t + y_s = -\frac{mg}{k} + C_1 \cos \sqrt{\frac{k}{m}}\, t + C_2 \sin \sqrt{\frac{k}{m}}\, t$$

Evaluating the constants C_1 and C_2 from the initial conditions we find that

$$C_1 = y_o + \frac{mg}{k}$$

$$C_2 = 0$$

The complete solution is

$$y = -\frac{mg}{k} + (y_o + \frac{mg}{k}) \cos \sqrt{\frac{k}{m}}\, t$$

Now let us calculate the kinetic and potential energy for the system. The kinetic energy is

$$T = \frac{1}{2} m \dot{y}^2 .$$

The potential energy is due to the work done against the force of gravity and also the spring force, both being conservative forces.

$$V = mg\, y + \frac{1}{2} k\, y^2$$

the reference point for zero potential energy being taken at $y = 0$. Since this is a conservative system, the total energy E is constant and equal to its initial value

$$E = T + V = mg\, y_o + \frac{1}{2} k\, y_o^2$$

It can be seen that the kinetic energy is maximum when the potential energy is minimum and vice versa. Therefore,

$$V_{max} = E$$

since

$$T_{min} = 0$$

On the other hand V_{min} is found to occur at

$$y = -\frac{mg}{k}$$

in which case

$$V_{min} = -\frac{1}{2} \frac{m^2 g^2}{k}$$

and therefore,

$$T_{max} = E + \frac{1}{2} \frac{m^2g^2}{k}$$

Thus we see that

$$V_{max} - V_{min} = T_{max} - T_{min}$$

even though the extreme values of V are not equal to the corresponding extreme values of T.

Now let us consider the case where the displacement is measured from its equilibrium position. Calling this vertical displacement z, we can write

$$z = y + \frac{mg}{k}$$

in which case the complete solution is

$$z = z_o \cos \frac{k}{m} t$$

where

$$z_o = y_o + \frac{mg}{k}$$

Now let us choose the zero reference for potential energy to be at z = 0, i. e.,

$$V = mgz + \frac{1}{2} k y^2 - \frac{1}{2} \frac{m^2g^2}{k} = mgz + \frac{1}{2} k(z - \frac{mg}{k})^2 - \frac{1}{2} \frac{m^2g^2}{k}$$

which reduces to

$$V = \frac{1}{2} kz^2$$

Therefore, if we use the static equilibrium position as the zero reference for potential energy, we find that the total energy is

$$E = T + V = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} k z^2$$
$$= \frac{1}{2} kz_o^2$$

Also

$$T_{max} = V_{max} = E$$

and

$$T_{min} = V_{min} = 0$$

In summary, it can be seen that the analysis is simplified by measuring displacements from the position of static equilibrium and setting the potential energy equal to zero at this point. In this case the total force on the particle is

$$-\frac{\partial V}{\partial z} = -kz$$

giving

$$V = \frac{1}{2}\,kz^2$$

as found above. Note that this includes gravitational as well as elastic potential energy.

Example 3-3. A particle of mass m starts from rest and slides on a frictionless track around a vertical circular loop of radius r. Find the minimum starting height h above the bottom of the loop in order that the particle will not leave the track at any point.

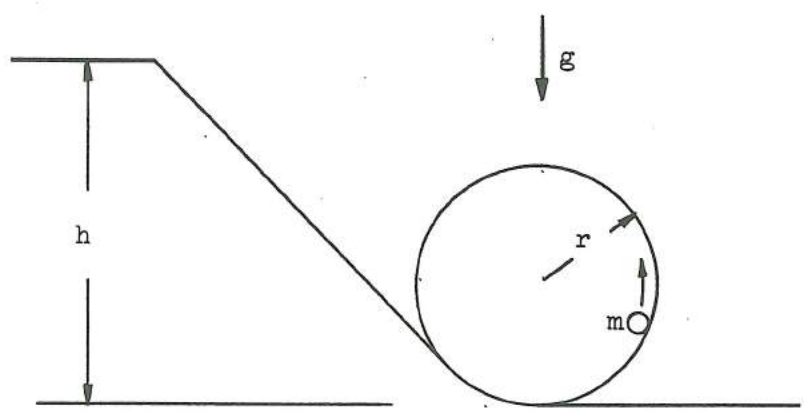

Figure 3-10

In this example, the track is fixed and frictionless so it does no work on the particle. Therefore, the only external force that works on the particle is the force of gravity. So we can use the principle of conservation of energy to find the speed of the particle at any point.

Choose the zero reference for potential energy at the bottom of the track. Clearly, the minimum speed during the loop occurs at the top and is found from

$$mgh = 2\,mgr + \frac{m}{2}\,v^2$$

giving

$$v^2 = 2g(h - 2r)$$

Now if the particle velocity is such that it just avoids leaving the track at the top of the loop, then the force of track on the particle is zero at this point but at no other point. So the gravitational force is equal to the mass times the centripetal acceleration at this point.

$$mg = m\,\frac{v^2}{r}$$

or

$$v^2 = rg$$

Equating the two expressions for v^2 given above, we obtain

$$h_{min} = \frac{5}{2}\,r$$

Example 3-4. A particle of mass m slides along a frictionless horizontal track in the form of a logarithmic spiral

$$r = r_o\,e^{-a\,\theta}$$

If its initial speed is v_o when $\theta = 0$, find its speed and also the magnitude of the track force as a function of position.

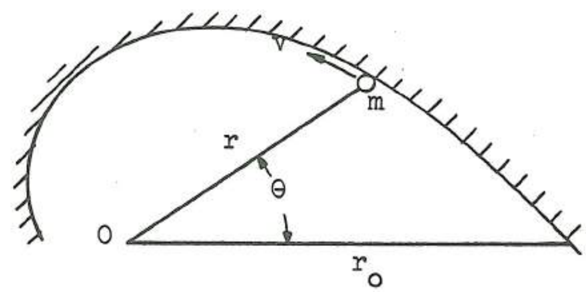

Figure 3-11

In this case the only force acting on the particle is the track force normal to the direction of motion. Therefore, from Equation (3-69) the speed along the track is constant

$$v = v_o$$

From the equation of the curve we find that

$$\frac{\dot{r}}{r\dot{\theta}} = -a$$

implying that the ratio of radial to tangential velocities remains constant, and therefore, the angle is constant between the curve and a radial line from point 0. Let this angle be α where

$$\tan \alpha = \frac{1}{a}$$

as shown in Figure 3-12.

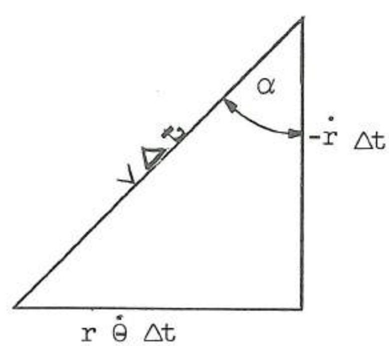

Figure 3-12

The track force will be calculated from the rate of change of angular momentum about 0.

$$H = m\ r\ v \sin \alpha$$

$$\dot{H} = m\ \dot{r}\ v \sin \alpha = -m\ v^2 \sin \alpha \cos \alpha$$

From Equation (3-76) we see that the moment of the track force about 0 must be constant, since $\dot{H}$ is constant, and is equal to

$$-F\ r \cos \alpha \quad = -m\ v^2 \sin \alpha \cos \alpha$$

from which we obtain

$$F = \frac{mv_o^2}{r} \sin \alpha = \frac{mv_o^2}{r\sqrt{1 + a^2}}$$

or

$$F = \frac{mv_o^2 e^{a\theta}}{r_o \sqrt{1 + a^2}}$$

Example 3-5. A particle of mass m is projected horizontally with velocity v_m in the direction of the positive y-axis onto a horizontal belt that is moving with a uniform velocity v_b in the direction of the positive x-axis, as shown in Figure 3-13. There is a coefficient of sliding friction μ between the particle and the belt. Assuming that the particle first touches the belt at the origin of the fixed xy coordinate system and remains on the belt, find the coordinates (x,y) of the point where sliding stops.

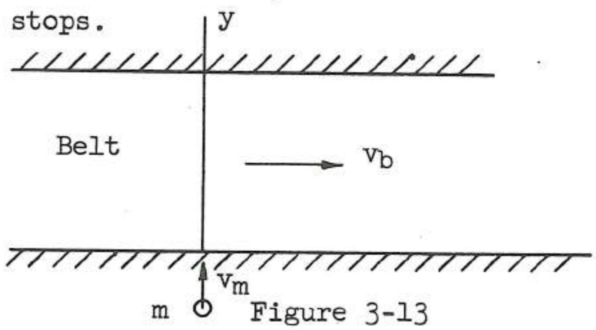

Figure 3-13

Before proceeding with the solution of the problem, consider first the nature of the force of Coulomb or sliding friction. Suppose two bodies A and B are sliding relative to each other and the force transmitted by the flat contact surface has a component N normal to that surface (Figure 3-14). Then the force of friction acting on A is

$$\overline{F}_f = -\frac{\mu N}{v_r}\overline{v}_r \qquad (3\text{-}84)$$

where $\overline{v}_r$ is the velocity of A relative to B at the contact surface. Of course, the frictional force on B is equal and opposite. It can be seen that the force of friction is always in a direction opposite to $\overline{v}_r$

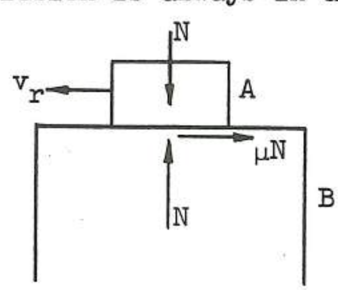

Figure 3-14

but is independent of its magnitude. Also, it is independent of the area of contact. (In a more general sense, one can think of the force of sliding friction as resulting from a frictional shear stress at the contact area that is equal to μ times the normal pressure. This would give the same result in this case, but would aid in the analysis of more general cases such as those with a curved contact area or nonuniform velocity or pressure distributions.)

It should be noted in passing that the force required to initiate sliding is somewhat larger than the force required to sustain it. The equation for the magnitude of this force is

$$F_s = \mu_s N \tag{3-85}$$

where μ_s is the coefficient of static friction.

$$\mu_s > \mu \tag{3-86}$$

Returning to the problem at hand, first note that the fixed xy coordinate system and also a reference frame moving with the belt are both inertial systems since the belt moves with uniform velocity. Because of frictional force depends upon the motion of the particle relative to the belt, it is more convenient to consider the motion relative to a coordinate system $x'y'$ that is fixed in the belt and moves with it. As viewed by an observer riding with the belt, the particle moves onto the belt with a velocity component v_m in the positive y' direction and a component v_b in the negative x' direction. The path of the particle relative to the belt is a straight line since the frictional force directly opposes the motion, and there are no horizontal forces perpendicular to the path.

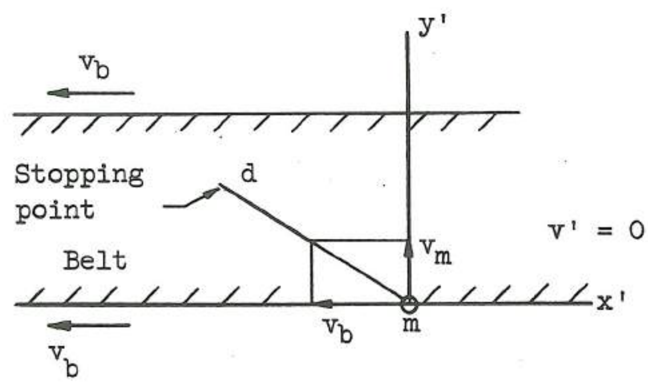

Figure 3-15

Let us use the principle of work and kinetic energy to find the stopping point in the x'y' frame. The work done against friction is equal to the loss of kinetic energy, or

$$\mu\, mg\, d = \frac{m}{2}\,(v_m^2 + v_b^2)$$

where d is the stopping distance, and the normal force is mg. Solving for the stopping distance

$$d = \frac{1}{2\mu g}\,(v_m^2 + v_b^2)$$

Assuming that the particle moved onto the belt at the origin of the x'y' system, the stopping point is

$$x' = -d\,\frac{v_b}{\sqrt{v_m^2 + v_b^2}} = \frac{-v_b}{2\mu g}\,\sqrt{v_m^2 + v_b^2}$$

$$y' = d\,\frac{v_m}{\sqrt{v_m^2 + v_b^2}} = \frac{v_m}{2\mu g}\,\sqrt{v_m^2 + v_b^2}$$

The time required to stop is just the original speed divided by the constant deceleration μg, or

$$t = \frac{1}{\mu g}\,\sqrt{v_m^2 + v_b^2}$$

Now let us convert back to the fixed xy coordinate system by means of the equations

$$x = x' + v_b\, t$$

$$y = y'$$

from which we obtain the coordinates in fixed space of the point where sliding stops.

$$x = \frac{v_b}{2\mu g}\,\sqrt{v_m^2 + v_b^2}$$

$$y = \frac{v_m}{2\mu g}\,\sqrt{v_m^2 + v_b^2}$$

This result could also have been obtained by working entirely in the fixed xy frame. Again it is seen that the frictional force is constant in magnitude and direction and so is the resulting acceleration μg which is in a direction opposite to the velocity of the particle relative to the belt. Taking components of acceleration in the x and y directions,

$$a_x = \mu g \frac{v_b}{\sqrt{v_m^2 + v_b^2}}$$

$$a_y = -\mu g \frac{v_m}{\sqrt{v_m^2 + v_b^2}}$$

The time required for the y velocity to reach zero is just

$$t = -\frac{v_m}{a_y} = \frac{1}{\mu g} \sqrt{v_m^2 + v_b^2}$$

at which time the x component of velocity is

$$v_x = a_x\, t = v_b$$

which is just the belt velocity, indicating that the sliding stops in the x and y directions simultaneously. The displacement at this time is found from Equation (3-7) to be

$$x = -\, a_x\, t^2 = \frac{v_b}{2\mu g} \sqrt{v_m^2 + v_b^2}$$

$$y = v_m\, t + \tfrac{1}{2} a_y\, t^2 = \frac{v_m}{2\mu g} \sqrt{v_m^2 + v_b^2}$$

in agreement with our previous results. The path of the particle in fixed space is parabolic while sliding occurs, as shown in Figure 3-16,

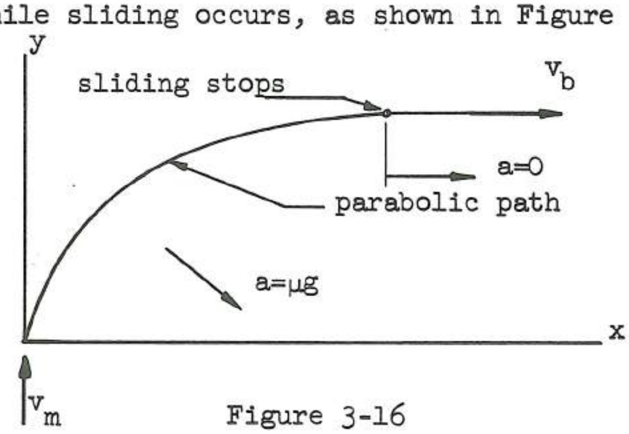

Figure 3-16

CHAPTER 4

DYNAMICS OF A SYSTEM OF PARTICLES

4-1. Equations of Motion

In Chapter 3, we obtained some of the important principles and techniques to be used in the analysis of the motion of a single particle. In this chapter, we shall study the motion of a system of particles by extending the principles previously derived and by introducing others.

Consider first a system of n particles, of which three are shown in Figure 4-1. A given particle may have both external and internal forces applied to it. The total force on the ith particle arising from sources external to the system of n particles is designated by $\overline{F}_i$ and is known as an <u>external force</u>. All interaction forces among the particles are known as <u>internal forces</u> and are designated by individual force vectors of the form $\overline{f}_{ij}$, where the first subscript indicates the particle on which the force acts and the second subscript indicates the acting particle.

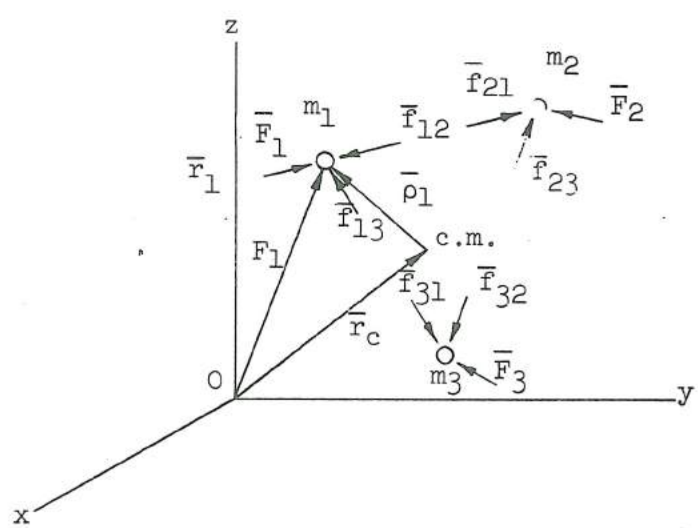

Figure 4-1

We know from Newton's law of action and reaction that the interaction forces between any two particles are equal and opposite, i.e.,

$$\overline{f}_{ij} = -\overline{f}_{ji} \tag{4-1}$$

and also that they act along the same straight line connecting the particles.

Now consider the forces acting on the ith particle. Including both external and internal forces, we can write the equation of motion.

$$m_i\ddot{\bar{r}}_i = \bar{F}_i + \sum_{j=1}^{n} \bar{f}_{ij} \quad (i = 1, 2, \ldots, n) \tag{4-2}$$

where

$$\bar{f}_{ii} = 0$$

and where m_i is the mass of the ith particle and $\bar{r}_i$ is its position vector relative to a fixed point O.

Summing Equation (4-2) over all particles and using Equation (4-1), we obtain

$$\sum_{i=1}^{n} m_i\ddot{\bar{r}}_i = \sum_{i=1}^{n} \bar{F}_i \tag{4-3}$$

But the total mass is

$$m = \sum_{i=1}^{n} m_i \tag{4-4}$$

and the location of the center of mass is

$$\bar{r}_c = \frac{1}{m}\sum_{i=1}^{n} m_i\bar{r}_i \tag{4-5}$$

So Equation (4-3) can be written in the form

$$\bar{F} = m\,\ddot{\bar{r}}_c \tag{4-6}$$

where

$$\bar{F} = \sum_{i=1}^{n} \bar{F}_i \tag{4-7}$$

In other words, the motion of the center of mass of a system of particles is the same as if the entire mass of the system were concentrated at the center of mass and were driven by the resultant of all forces external to the system.

4-2. Work and Kinetic Energy

Because of the similarity in form between Equation (4-6) and the equation of motion for a single particle, $\overline{F} = m\ddot{\overline{r}}$, it is apparent that a set of principles similar to those for a single particle also applies to the motion of the center of mass of a system of particles.

Let us assume that the center of mass of the system moves from A_c to B_c under the action of the forces acting on the system. Then, taking the line integral of each side of Equation (4-6) in a manner similar to that shown in Figure 3-1, we obtain a result similar to Equation (3-35).

$$\int_{A_c}^{B_c} \overline{F}\cdot d\overline{r}_c = \frac{m}{2} v_c^2 \Big|_{A_c}^{B_c} \tag{4-8}$$

This equation states that if one considers the resultant of the external forces to be acting at the center of mass of the system, then the work done by the external forces in moving over the path of the center of mass is equal to the change in the translational kinetic energy of the total mass as it moves along that path.

The work expressed by the integral on the left side of Equation (4-8) does not include that work done by the internal forces, nor is it even the total work of the external forces. To see this more clearly, let us calculate the total work. The total work done by all the forces acting on the ith particle as it moves from A_i to B_i is

$$W_i = \int_{A_i}^{B_i} (\overline{F}_i + \sum_{j=1}^{n} \overline{f}_{ij})\cdot d\overline{r}_i \tag{4-9}$$

But we can write the position vector of the ith particle as the sum

$$\overline{r}_i = \overline{r}_c + \overline{\rho}_i \tag{4-10}$$

where $\overline{\rho}_i$ is the position vector of the ith particle relative to the center of mass, as shown in Figure 4-1. So, summing over all particles, using Equations (4-9) and (4-10), we obtain the total work

$$W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \int_{A_i}^{B_i} (\overline{F}_i + \sum_{j=1}^{n} \overline{f}_{ij})\cdot(d\overline{r}_c + d\overline{\rho}_i)$$

or

$$W = \sum_{i=1}^{n} \int_{A_c}^{B_c} (\overline{F}_i + \sum_{j=1}^{n} \overline{f}_{ij}) \cdot d\overline{r}_c + \sum_{i=1}^{n} \int_{A_i}^{B_c} (\overline{F}_i + \sum_{j=1}^{n} \overline{f}_{ij}) \cdot d\overline{\rho}_i \qquad (4\text{-}11)$$

where the limits on the second integral refer to the position of the ith particle relative to the position of the center of mass of the system.

From Equation (4-1) we see that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{f}_{ij} = 0 \qquad (4\text{-}12)$$

Using this equation and also Equation (4-7) we can simplify Equation (4-11) for the total work.

$$W = \int_{A_c}^{B_c} \overline{F} \cdot d\overline{r}_c + \sum_{i=1}^{n} \int_{A_i}^{B_i} (\overline{F}_i + \Sigma\, \overline{f}_{ij}) \cdot d\overline{\rho}_i \qquad (4\text{-}13)$$

Thus the total work is the work done by the total external force acting through the displacement of the center of mass plus the work done by the external and internal forces on each particle acting through the displacement of the particle relative to the center of mass.

Now, for each particle, the equation of work and kinetic energy applies, so

$$W_i = \frac{m_i}{2} \left[\dot{\overline{r}}_i \cdot \dot{\overline{r}}_i \right]_{A_i}^{B_i} = \frac{m_i}{2} \left[\dot{\overline{r}}_c \cdot \dot{\overline{r}}_c + 2\dot{\overline{r}}_c \cdot \dot{\overline{\rho}}_i + \dot{\overline{\rho}}_i \cdot \dot{\overline{\rho}}_i \right]_{A_i}^{B_i} \qquad (4\text{-}14)$$

where we have substituted for $\overline{r}_i$ according to Equation (4-10). Summing Equation (4-14) over all particles and noting that

$$\sum_{i=1}^{n} m_i \overline{\rho}_i = 0 \qquad (4\text{-}15)$$

because $\overline{\rho}_i$ is measured from the center of mass, we obtain

$$W = \frac{m}{2} v_c^2 \Big|_{A_c}^{B_c} + \sum_{i=1}^{n} \frac{m_i}{2} v_i^2 \Big|_{A_i}^{B_i} \qquad (4\text{-}16)$$

where v_i is the speed of the ith particle as viewed by an observer translating with the center of mass. The right side of the above equation represents the change in the total kinetic energy.

$$W = T_B - T_A \tag{4-17}$$

where T_A and T_B are the total kinetic energy of the system at the beginning and end of the line integrations, respectively.

$$T = \frac{m}{2} v_c^2 + \sum_{i=1}^{n} \frac{m_i}{2} v_i^2 \tag{4-18}$$

It can be seen that, if finite external and internal forces act upon the system, then as the time interval being considered approaches zero, the changes in kinetic energy and the work also approach zero but remain equal in accordance with Equation (4-16). In the limit we can write.

$$\dot{W} = \dot{T} \tag{4-19}$$

i.e., the total rate at which both external and internal forces do work on a system of particles is equal to its rate of increase of kinetic energy.

Returning now to Equations (4-8), (4-13), and (4-16), we find that

$$\sum_{i=1}^{n} \int_{A_i}^{B_i} (\overline{F}_i + \overline{f}_{ij}) \cdot d\overline{\rho}_i = \sum \frac{m_i}{2} v_i^2 \Big|_{A_i}^{B_i} \tag{4-20}$$

implying that the change in total kinetic energy due to particle velocities relative to the center of mass is equal to the work done by the external and internal forces as each particle moves through its displacement relative to the center of mass.

We can summarize the results of this section as follows:

1. The total kinetic energy is equal to that due to the total mass moving with the velocity of the center of mass plus that due to the motions of the individual particles relative to the center of mass.

2. The work done by the external forces in moving through the displacement of the center of mass is equal to the change in kinetic energy of the total mass moving with the speed of the center of mass.

3. The work done by the external plus internal forces acting through the relative displacements is equal to the change in the kinetic energy due to motion of the particles relative to the mass center. This kinetic energy is a result of the relative motion due to changing particle distances as well as that due to rigid body rotation in which the particle separations do not change with time.

4-3. Conservation of Mechanical Energy

The principle of conservation of mechanical energy was developed for the case of a single particle in Section 3-3. This principle can be extended to systems of particles in a direct manner.

First recall from Equation (4-6) that the motion of the center of mass of the system is the same as though the total external force were acting on the total mass concentrated at the mass center. Therefore, if the external forces are conservative, i.e., if they are derivable from a potential function involving coordinates only, then the center of mass moves such that

$$T_c + V_c = E_c \tag{4-21}$$

is a constant, where

T_c = kinetic energy due to translational motion of the center of mass

V_c = potential energy associated with the external forces acting on the system.

Equation (4-21) is valid even though the internal forces may be dissipative since, as we have seen, E_c is not the total energy but just a portion of it.

For the case where the internal as well as external forces are conservative we can write

$$T + V = E \tag{4-22}$$

where, in this case, the total energy is conserved. The total kinetic energy T is simply the sum of the kinetic energies of the individual particles. The total potential energy V is normally the sum of the potential energy due to gravity and that due to deformation of elastic elements in the system such as springs. In any event the potential energy is of the form

$$V = V(x_1, x_2, x_3, \ldots\ldots x_{3n}) \tag{4-23}$$

for the case of n particles moving in three-dimensional space. Then, if a small increase in x_m results in the small displacement of a certain particle in a given direction, the force

$$F_m = - \frac{\partial V}{\partial x_m} \qquad (4\text{-}24)$$

acts on the given particle in the direction of increasing x_m.

The results of Equations (4-23) and (4-24) apply to the potential energy V_c, as well, except that the forces obtained are external forces only. A particularly simple case results when the system moves in a uniform gravitational field, in which case the potential energy is a function of just the center of mass position, and is not dependent upon the specific locations of the individual particles. Then the center of mass moves as though it were a single particle being acted upon by the gravitational field.

4-4. Impulse and Momentum

In a manner similar to that used in obtaining the equation of impulse and momentum for a single particle, Equation (4-6) can be integrated with respect to time to give

$$\int_{t_1}^{t_2} \overline{F}\, dt = m(\overline{v}_{c2} - \overline{v}_{c1}) \qquad (4\text{-}25)$$

where v_{c2} and v_{c1} are the velocities of the center of mass at times t_2 and t_1, respectively. The integral on the left is the total impulse of the external forces during the given interval and the right side represents the change in total linear momentum of the system in the same interval. Note that because of Equation (4-1), the total impulse of the internal forces is zero, and they have no influence on the total linear momentum of the system.

Equation (4-25) could have been written in terms of Cartesian components as follows:

$$\mathcal{F}_x = \int_{t_1}^{t_2} F_x \, dt = m(\dot{x}_{c2} - \dot{x}_{c1})$$

$$\mathcal{F}_y = \int_{t_1}^{t_2} F_y \, dt = m(\dot{y}_{c2} - \dot{y}_{c1}) \qquad (4\text{-}26)$$

$$\mathcal{F}_z = \int_{t_1}^{t_2} F_z \, dt = m(\dot{z}_{c2} - \dot{z}_{c1})$$

where $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{F}_z$ are the components of the total impulse $\mathcal{F}$ due to external forces.

It can be seen that if any component of the total impulse is zero, then the momentum is conserved in this direction. Furthermore, if there are no external forces acting on the system, regardless of the nature of the internal forces, then the total momentum is constant. This is principle of conservation of linear momentum as it applies to a system of particles. It is particularly useful in the analysis of problems in which the internal forces are unknown such as in collision and explosion problems.

4-5. Angular Momentum

In Section 3-6, equations were developed for the angular momentum of a single particle and also its time rate of change, using as a reference a point fixed in inertial space. In this section we will extend this development to cover a system of particles, and in doing so we will consider several possible reference points.

Fixed reference point. Let us consider the total angular momentum of a system of n particles (Figure 4-1), taking as a reference the fixed point O. Using Equation (3-72) we see that the angular momentum of the ith particle is

$$\overline{H}_i = \overline{r}_i \times m_i \dot{\overline{r}}_i \qquad (4\text{-}27)$$

The total angular momentum of the system is merely the vector sum of the angular momenta of the individual particles.

$$\overline{H} = \sum_{i=1}^{n} \overline{H}_i = \sum_{i=1}^{n} \overline{r}_i \times m_i \dot{\overline{r}}_i \qquad (4\text{-}28)$$

Also,

$$\dot{\overline{H}} = \sum_{i=1}^{n} \overline{r}_i \times m_i \ddot{\overline{r}}_i \qquad (4\text{-}29)$$

Using Equation (4-2) the right side of Equation (4-29) can be changed to the form

$$\dot{\overline{H}} = \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i + \sum_{i=1}^{n} \sum_{i=1}^{n} \overline{r}_i \times \overline{f}_{ij} \tag{4-30}$$

But

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{r}_i \times \overline{f}_{ij} = 0 \tag{4-31}$$

since the internal forces occur in equal, opposite and collinear pairs. In other words, for every internal force $\overline{f}_{ij}$, there is another force $\overline{f}_{ji}$ whose moment about point 0 exactly cancels that due to $\overline{f}_{ij}$.

Therefore,

$$\dot{\overline{H}} = \overline{M} \tag{4-32}$$

where

$$\overline{M} = \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i \tag{4-33}$$

is the total moment about 0 of the external forces acting on the system.

Reference point at the center of mass. We can also write the angular momentum about 0 in terms of the position vector relative to the center of mass. From Equations (4-10) and (4-28) we obtain

$$\overline{H} = \sum_{i=1}^{n} (\overline{r}_c + \overline{\rho}_i) \times m_i(\dot{\overline{r}}_c + \dot{\overline{\rho}}_i) \tag{4-34}$$

We have seen from Equation (4-15) that

$$\sum_{i=1}^{n} m_i \overline{\rho}_i = 0$$

and

$$\sum_{i=1}^{n} m_i \dot{\overline{\rho}}_i = 0$$

so Equation (4-34) reduces to

$$\bar{H} = \bar{r}_c \times m\,\dot{\bar{r}}_c + \sum_{i=1}^{n} \bar{\rho}_i \times m_i\,\dot{\bar{\rho}}_i \qquad (4\text{-}35)$$

where m is the total mass. The first term of Equation (4-35) can be interpreted as the angular momentum due to all the particles moving at the velocity of the center of mass, or as the moment of momentum of a particle of mass m located at the center of mass and moving with it.

The second term of Equation (4-35) is the angular momentum of the system with respect to the center of mass, as viewed by a nonrotating observer moving with the center of mass. Let us call this term $\bar{H}_c$.

$$\bar{H}_c = \sum_{i=1}^{n} \bar{\rho}_i \times m_i\dot{\bar{\rho}}_i \qquad (4\text{-}36)$$

So we can summarize by observing that the total angular momentum about a fixed point O is the sum of (1) the angular momentum about O due to translational motion with the velocity of the center of mass and (2) the angular momentum of the system relative to the center of mass.

Now let us differentiate Equation (4-35), obtaining

$$\dot{\bar{H}} = \bar{r}_c \times m\,\ddot{\bar{r}}_c + \sum_{i=1}^{n} \bar{\rho}_i \times m_i\,\ddot{\bar{\rho}}_i \qquad (4\text{-}37)$$

We have seen in Equation (4-6) that

$$\bar{F} = m\,\ddot{\bar{r}}_c$$

Also, from Equation (4-36) we find that

$$\dot{\bar{H}}_c = \sum_{i=1}^{n} \bar{\rho}_i \times m_i\,\ddot{\bar{\rho}}_i \qquad (4\text{-}38)$$

Therefore, we can write Equation (4-37) in the form

$$\dot{\bar{H}} = \bar{r}_c \times \bar{F} + \dot{\bar{H}}_c \qquad (4\text{-}39)$$

But from Equations (4-7), (4-10), (4-32), and (4-33) we see that

$$\dot{\bar{H}} = \sum_{i=1}^{n} \bar{r}_i \times \bar{F}_i = \sum_{i=1}^{n} (\bar{r}_c + \bar{\rho}_i) \times \bar{F}_i$$

or

$$\dot{\overline{H}} = \overline{r}_c \times \overline{F} + \sum_{i=1}^{n} \overline{\rho}_i \times \overline{F}_i \qquad (4\text{-}40)$$

Finally, from Equations (4-39) and (4-40) we obtain

$$\dot{\overline{H}}_c = \overline{M}_c \qquad (4\text{-}41)$$

where

$$\overline{M}_c = \sum_{i=1}^{n} \overline{\rho}_i \times \overline{F}_i \qquad (4\text{-}42)$$

is the moment of the external forces about the center of mass.

Comparing Equations (4-32) and (4-41) we see that they are of identical form. So we can state that the time rate of change of angular momentum of a system of particles relative to a given point is equal to the moment about that point of the external forces acting on the system, providing that the chosen point is either (1) fixed in inertial space or (2) the center of mass of the system.

A further conclusion is that if the moment of the external forces about either of the above reference points is zero, then the corresponding angular momentum is constant. This is the principle of conservation of angular momentum as it applies to a system of particles.

Arbitrary reference point. Now let us consider the angular momentum of a system of particles relative to an arbitrary point P, i.e., as viewed by a nonrotating observer that is moving with point P.

Let the xyz frame of Figure 4-2 be fixed in inertial space, and let the point P move in an arbitrary manner relative to this frame. The position vectors drawn from the fixed point O to the center of mass, the point P, and a typical particle m_i are designated by $\overline{r}_c$, $\overline{r}_p$, and $\overline{r}_i$, respectively. The position vectors of the center of mass and the particle m_i relative to the point P are $\overline{\rho}_c$ and $\overline{\rho}_i$, respectively. It can be seen that

$$\overline{r}_i = \overline{r}_p + \overline{\rho}_i \qquad (4\text{-}43)$$

$$\overline{r}_c = \overline{r}_p + \overline{\rho}_c \qquad (4\text{-}44)$$

Also,

$$\overline{\rho}_c = \frac{1}{m} \sum_{i=1}^{n} m_i \, \overline{\rho}_i \qquad (4\text{-}45)$$

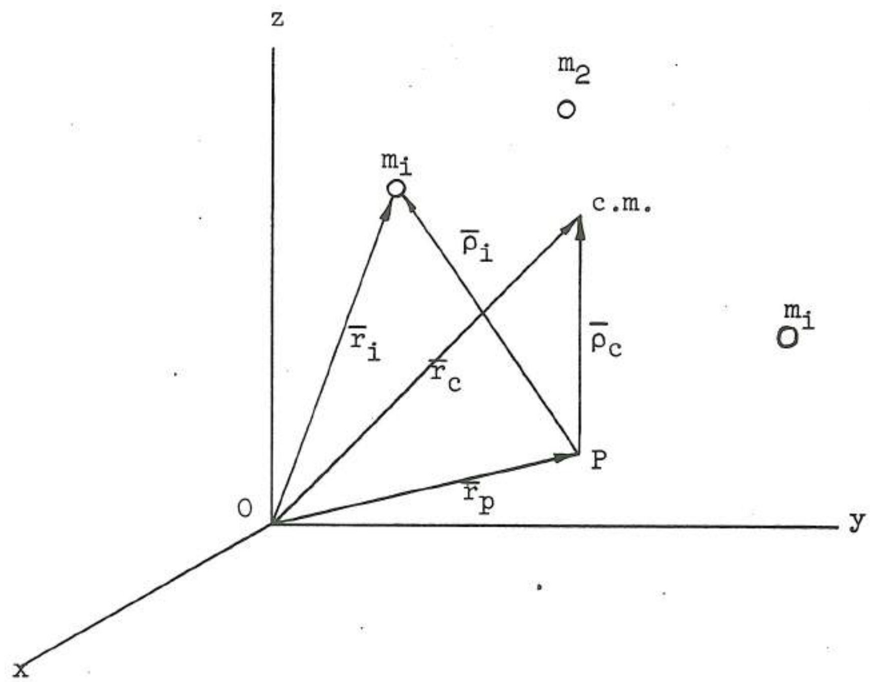

Figure 4-2

which is the defining equation for the location of the center of mass.

From Equations (4-28) and (4-43) we can write an expression for the angular momentum about O.

$$\bar{H} = \sum_{i=1}^{n} \bar{r}_i \times m_i \dot{\bar{r}}_i = \sum_{i=1}^{n} (\bar{r}_p + \bar{\rho}_i) \times m_i (\dot{\bar{r}}_p + \dot{\bar{\rho}}_i) \qquad (4\text{-}46)$$

which can be simplified using Equations (4-44) and (4-45) to give

$$\bar{H} = \bar{H}_p + \bar{r}_c \times m \dot{\bar{r}}_p + \bar{r}_p \times m \dot{\bar{\rho}}_c \qquad (4\text{-}47)$$

where $\bar{H}_p$ is the angular momentum of the system relative to P.

$$\bar{H}_p = \sum_{i=1}^{n} \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i \qquad (4\text{-}48)$$

Now let us differentiate Equation (4-47) with respect to time and use Equation (4-44) to obtain

$$\dot{\bar{H}} = \dot{\bar{H}}_p + (\dot{\bar{r}}_p + \dot{\bar{\rho}}_c) \times m \dot{\bar{r}}_p + (\bar{r}_p + \bar{\rho}_c) \times m \ddot{\bar{r}}_p + \dot{\bar{r}}_p \times m \dot{\bar{\rho}}_c + \bar{r}_p \times m \ddot{\bar{\rho}}_c$$

which reduces to

$$\dot{\overline{H}} = \dot{\overline{H}}_p + \overline{\rho}_c \times m\, \ddot{\overline{r}}_p + \overline{r}_p \times m\, \ddot{\overline{r}}_c \tag{4-49}$$

The moment about O of the external forces is found in the same manner as in Equation (4-40).

$$\overline{M} = \overline{r}_p \times \overline{F} + \sum_{i=1}^{n} \overline{\rho}_i \times \overline{F}_i \tag{4-50}$$

or

$$\overline{M} = \overline{r}_p \times \overline{F} + \overline{M}_p \tag{4-51}$$

where $\overline{M}_p$ is the moment about P of the external forces.

$$\overline{M}_p = \sum_{i=1}^{n} \overline{\rho}_i \times \overline{F}_i \tag{4.52}$$

Finally, from Equations (4-6), (4-32), (4-49), and (4-51), we obtain

$$\dot{\overline{H}}_p = \overline{M}_p - \overline{\rho}_c \times m\, \ddot{\overline{r}}_p \tag{4-53}$$

A comparison of Equation (4-53) with Equations (4-32) and (4-41) reveals that the choice of an arbitrary reference point has resulted in an additional term, $-\overline{\rho}_c \times m\, \ddot{\overline{r}}_p$. Of course, if P is fixed, then $\ddot{\overline{r}}_p = 0$ and the term is zero. If P is at the center of mass, then $\overline{\rho}_c = 0$ and again the term is zero, in agreement with our earlier results. If $\overline{\rho}_c$ and $\ddot{\overline{r}}_p$ are parallel, the term also disappears, but this situation is not very likely to occur over an extended interval of time.

The physical interpretation of this term is that it is the moment about P of the inertial or D'Alembert force, $-m\ddot{\overline{r}}_p$, due to the fact that that the nonrotating reference frame moving with P is not an inertial frame. This resultant inertial force is the sum of the individual inertial forces at the particles and it acts on a line through the center of mass. Thus, one could use the standard form of the equation, $\dot{\overline{H}} = \overline{M}$, even for the case of an accelerating reference point if he were to include inertial forces (due to $\ddot{\overline{r}}_p$) as well as the actual external forces in calculating $\overline{M}$.

The above interpretation is an application of the following general rule:

> All results and principles derivable from Newton's laws of motion relative to an inertial frame can be extended to apply to an accelerating but nonrotating frame if the

inertial forces due to the acceleration of the frame are considered as additional external forces acting on the system.

This rule applies to all calculations including those of work, kinetic energy, momentum, angular momentum, etc. It is seen that the original introduction of inertial forces in Equation (1-34) is the special case where the reference frame translates with the particle. Thus there is no motion relative to this reference frame and the situation reduces to that of static equilibrium with the true externalforces being balanced by inertial forces.

A word of caution should be given at this point. On the whole, it is advisable to consider dynamics problems from the viewpoint of an observer in an inertial frame of reference and, therefore, to include only the true external forces acting on a particle or set of particles when applying Newton's equation of motion. On the other hand, certain problems, or portions of an analysis, are made simpler by adopting the viewpoint of an accelerating observer. A common example of the latter is the choice of the center of mass (which may be accelerating) as the reference point for the analysis of the rotational aspects of the motion, as in Equation (4-41).

Equation (4-53), utilizing an arbitrary reference point P, also is very convenient for certain problems. One example is the case where the motion of a given point in the system is a known function of time. Choosing this point as the reference point P, one can immediately calculate the inertial force $-m\ddot{\bar{r}}_p$. Furthermore, the external force acting at P has no moment about P and thus need not be calculated, whereas it would have to be calculated if another reference point were chosen.

Another example is the writing of the rotational equations for a satellite or space vehicle with moving parts. It is convenient to choose a reference point that is fixed in one (normally the largest) part of the system and to specify the location of the various parts relative to this reference, as they would naturally appear to an observer on the vehicle. This is in contrast to having to recompute the locations and moments of inertia of all parts relative to the center of mass as the center of mass moves, and this may occur due to the relative motion of a single part.

4-6. Angular Impulse

The equation relating the angular impulse due to external forces and the change of angular momentum can be derived for the case of a system of particles in a manner similar to that used in obtaining Equation (3-83) for a single particle.

We have seen from Equations (4-32) and (4-41) that

$$\dot{\bar{H}} = \bar{M}$$

applies to a system of particles when the reference point is either fixed or is at the center of mass. Integration of both sides of this equation with respect to time over the interval from t_1 to t_2 results in

$$\overline{\mathcal{M}} = \overline{H}_2 - \overline{H}_1 \qquad (4\text{-}54)$$

where

$$\overline{\mathcal{M}} = \int_{t_1}^{t_2} \overline{M}\, dt \qquad (4\text{-}55)$$

is the total impulse acting on the system due to external forces. As we have seen, the internal forces between any two particles occur in equal, opposite, and collinear pairs, and therefore they cancel out in this calculation.

If we perform a similar integration on Equation (4-53) we obtain

$$\overline{H}_{p2} - \overline{H}_{p1} = \overline{\mathcal{M}}_p - \int_{t_1}^{t_2} \overline{\rho}_c \times m\, \ddot{\overline{r}}_p\, dt \qquad (4\text{-}56)$$

where the integral on the right can be interpreted as the angular impulse about P of the inertial forces arising from the acceleration of P.

4.7 Collisions

In order to clarify some of the important features of collision problems, let us first consider the special case of two colliding spheres. The spheres will be assumed to be perfectly smooth, implying that all forces on a sphere pass through the center of mass, thereby reducing the problem to one of particle motion. Furthermore, let us assume that the spheres move in the same plane before impact, i.e., the velocity vectors and the line of centers at impact all lie in the same plane.

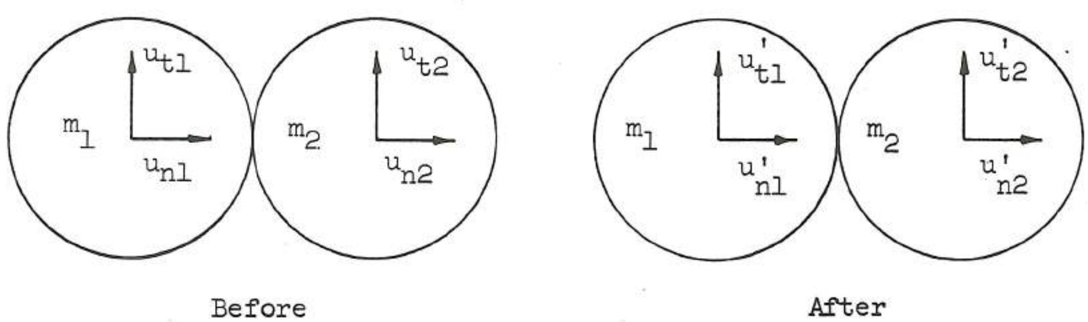

Figure 4-3

The situation is shown in Figure 4-3 where u_n represents a velocity component along the line of centers and u_t represents a velocity component perpendicular to the line of centers and in the plane of motion. The velocities are relative to an inertial frame.

An important assumption in the solution of collision problems is that the forces between the colliding bodies at the moment of impact are impulsive in nature, that is, the forces are very large compared to any other forces acting at the same time and are of very short duration. Thus they are best defined in terms of their total impulse rather than making any attempt to give their variation with time. Let us designate this impulse by $\overline{\mathcal{F}}$ in the present case. It acts to the left on m_1 and to the right on m_2, in each case being normal to the common tangent plane at the point of contact. Using the equation of linear impulse and momentum on each particle we can write

$$m_1 (u_{n1} - u'_{n1}) = \overline{\mathcal{F}} = m_2 (u'_{n2} - u_{n2})$$

or

$$m_1 u_{n1} + m_2 u_{n2} = m_1 u'_{n1} + m_2 u'_{n2} \qquad (4\text{-}57)$$

There is no impulse on the spheres in the tangential direction so the velocity components in this direction are unchanged.

$$u_{t1} = u'_{t1}$$

$$u_{t2} = u'_{t2} \qquad (4\text{-}58)$$

from which we obtain

$$m_1 u_{t1} + m_2 u_{t2} = m_1 u'_{t1} + m_2 u'_{t2} \qquad (4\text{-}59)$$

Equations (4-57) and (4-59) indicate that the components of linear momentum in the normal and tangential directions are conserved during impact, and thus the total linear momentum of the system is conserved. This is to be expected since no external forces are acting on the system.

But even if other forces do act on the system, Equations (4-57), (4-58), and (4-59) still apply, providing that these other forces are not impulsive forces applied at the instant of impact. The reason is that any non-impulsive forces applied over the infinitesimal time interval will have a negligible total impulse and thus a negligible effect in this interval.

In order to solve for the motion of the system, we must know the tangential and normal components of velocity after impact. The tangential components are given by Equation (4-58). However, another equation is required in addition to Equation (4-57) in order to be able to solve for both u'_{n1} and u'_{n2}. This equation relates the normal components of the velocities of approach and separation.

$$u'_{n2} - u'_{n1} = e(u_{n1} - u_{n2}) \quad (0 \leqslant e \leqslant 1) \tag{4-60}$$

where e is the coefficient of restitution. From Equations (4-57) and (4-60) we obtain

$$u'_{n1} = \frac{m_1 - e\, m_2}{m_1 + m_2}\, u_{n1} + \frac{(1 + e)m_2}{m_1 + m_2}\, u_{n2}$$

$$u'_{n2} = \frac{(1 + e)m_1}{m_1 + m_2}\, u_{n1} + \frac{m_2 - e\, m_1}{m_1 + m_2}\, u_{n2} \tag{4-61}$$

When $e = 0$, the normal velocities after impact are equal ($u'_{n1} = u'_{n2}$) and the collision is said to be inelastic. In this case, the system always loses kinetic energy during impact.

When $e = 1$, the relative velocities of approach and separation have the same magnitude and the collision is said to be perfectly elastic. In this case the total kinetic energy is conserved as well as the momentum. To see this most easily, let us view the collision from an inertial frame moving with the velocity of one of the spheres (m_1, for example) just before impact. Then $u_{n1} = u_{t1} = 0$, and we find, using Equation (4-61) that

$$\frac{1}{2} m_1 u'^2_{n1} + \frac{1}{2} m_2 u'^2_{n2} = \frac{1}{2} m_2 u^2_{n2} \tag{4-62}$$

Combining Equations (4-58) and (4-62), we obtain

$$\frac{1}{2} m_1 u'^2_{n1} + \frac{1}{2} m_2(u'^2_{n2} + u'^2_{t2}) = \frac{1}{2} m_2(u^2_{n2} + u^2_{t2}) \tag{4-63}$$

implying that the total kinetic energy is conserved. Since energy conservation does not depend upon which inertial frame is chosen as a reference, the kinetic energy is conserved in the original, fixed frame as well.

For the particular case of the perfectly elastic impact of smooth spheres of equal mass, the normal components of velocity of the spheres are interchanged, as can be seen from Equation (4-61).

$$u_{n1}' = u_{n2}$$

$$u_{n2}' = u_{n1} \qquad (4\text{-}64)$$

In general, we conclude that linear momentum is always conserved at the moment of impact, as indicated by Equations (4-57) and (4-59). On the other hand, mechanical energy is conserved only for the case of perfectly elastic impact ($e = 1$). The lost kinetic energy for $e \neq 1$ appears as heat or, in essence, as the kinetic energy of internal vibrations whose scale is relatively microscopic and therefore is not included in our analysis.

An interesting way of obtaining a better understanding of the physical nature of the coefficient of restitution is by considering the collision process to occur in two phases: (1) the compression phase during which the relative normal velocity of the spheres is reduced to zero, and (2) the restitution phase which lasts from the end of (1) until the spheres separate. Now a portion of the total collision impulse $\bar{\mathcal{F}}$ will occur during the compression phase and the remainder occurs during the restitution phase. Calling these impulses $\bar{\mathcal{F}}_c$ and $\bar{\mathcal{F}}_r$, respectively, we have

$$\bar{\mathcal{F}} = \bar{\mathcal{F}} + \bar{\mathcal{F}} \qquad (4\text{-}65)$$

All impulses on a given sphere occur in the same direction, so we need to consider only their magnitudes. Furthermore, since the change in velocity of a given mass is directly proportional to the applied impulse, we conclude from Equation (4-60) that

$$\mathcal{F}_r = e\,\mathcal{F}_c \qquad (4\text{-}66)$$

We assumed originally that both spheres move in the same plane before impact and the results of the analysis indicate that no forces are produced during the collision to cause the motion to deviate from that plane. Upon further study, however, it becomes apparent that this assumption was not really necessary and that the above results are valid for the more general case. This is because we are interested only in the forces and changes of velocity at the moment of impact. We have seen that all forces except the impulse $\bar{\mathcal{F}}$ can be neglected at this moment. Furthermore the collision can be observed from a reference frame translating at the velocity one of the spheres just before impact. From this viewpoint it can be seen that the important forces and velocities lie in the plane containing the line of centers at impact and the relative velocity vector.

4-8. The Rocket Problem

Particle mechanics approach. An important application of the principles of dynamics of a system of particles is in the analysis of rocket propulsion, i.e., propulsion by means of reaction forces due to the ejection of mass. As a simplified example, consider the rocket shown in Figure 4-4. Assume that the rocket is operating in a vacuum in the absence of gravitational forces. Let the

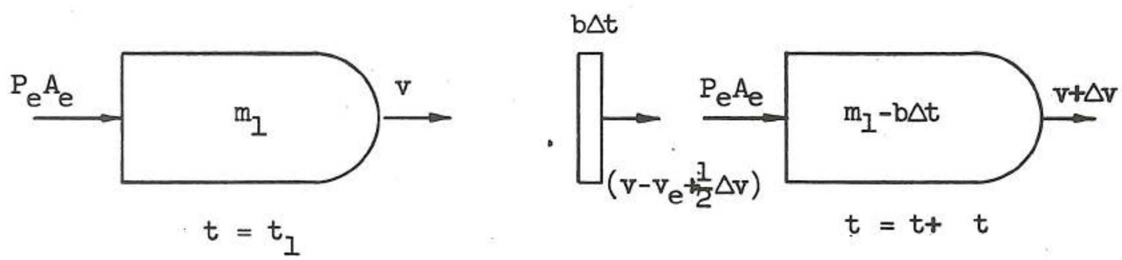

Figure 4-4

area of the nozzle exit be A_e and let the average pressure of the exhaust gases at this area be P_e. The average exit velocity of the exhause gases relative to the rocket is v_e. Assume also that the mass of the unburned fuel plus the rocket structure and payload is given by

$$m = m_o - bt \tag{4-67}$$

where b is the rate at which fuel is burned and ejected from the rocket. The parameter b is normally assumed to be constant during burning, but, in general,

$$b = \rho_e A_e v_e \tag{4-68}$$

where ρ_e is the average density of the discharged gases at the exit area A_e.

Let us take as the system of particles under consideration the total mass m_1 of the rocket at time t_1. We will consider the same system again after an infinitesimal interval Δt and use the equation of impulse and momentum to calculate the change in rocket velocity. The total impulse acting on the system during this interval is

$$\mathcal{F} = p_e \, A_e \, \Delta t \tag{4-69}$$

since the force due to the pressure p_e is the only external force on the system. The total momentum at $t = t_1 + \Delta t$ is

$$(m_1 - b\Delta t)\,(v + \Delta v) + b\Delta t(v + \tfrac{1}{2}\Delta v - v_e)$$

where the first term is the momentum of the rocket and unburned fuel and the second term is the momentum of the mass $b\Delta t$ ejected with velocity v_e in a rearward direction relative to the average rocket velocity $(v + \frac{1}{2}\Delta v)$ during the interval. The total momentum at $t = t_1$ is simply $m_1 v$. Therefore, equating the total impulse on the system to the change in linear momentum, we obtain

$$p_e A_e \Delta t = (m_1 - b\Delta t)(v + \Delta v) + b\Delta t(v + \frac{1}{2}\Delta v - v_e) - m_1 v$$

which simplifies to

$$p_e A_e \Delta t = m_1 \Delta v - b\, v_e \Delta t - \frac{1}{2} b \Delta v \Delta t \tag{4-70}$$

Diving by Δt and taking the limit as Δt approaches zero we find that

$$m\,\dot{v} = p_e A_e + b\, v_e \tag{4-71}$$

The last term of Equation (4-70) can be neglected as Δt approaches zero since Δv also approaches zero.

Equation (4-71) can also be written in the form

$$F_s = m\,\dot{v} \tag{4-72}$$

where F_s is the static thrust of the rocket

$$F_s = p_e A_e + b\, v_e \tag{4-73}$$

To see this more clearly, suppose the rocket of Figure 4-4 is held fixed by a test stand, as shown in Figure 4-5. The static thrust F_s is the force that is transmitted by the test stand to the earth and is also the force on the rocket that is required to keep it stationary. In this

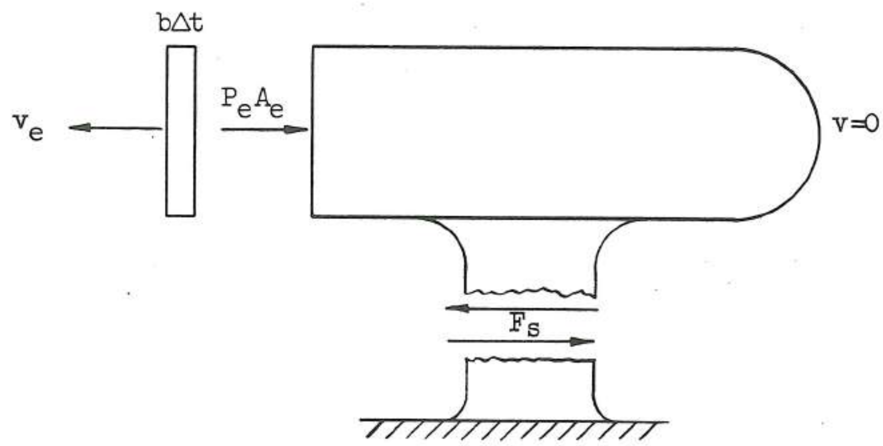

Figure 4-5

case, the total external force on the system is $(F_s - p_eA_e)$ in a rearward direction. The rocket itself has no momentum; the only momentum is that of the exhaust gases. So using impulse and momentum considerations,

$$(F_s - p_eA_e)\Delta t = b\ v_e\Delta t$$

or

$$F_s = p_eA_e + b\ v_e$$

in agreement with Equation (4-73). Note that we have again been careful to include the same particles in the system at the beginning and end of the interval Δt.

Now let us consider the forces acting on the unburned mass m when the test stand has been removed. In this case, the total force is the sum of the pressure force P_eA_e and the jet reaction force $b\ v_e$. In other words, it is just the static thrust F_s. From Equation (4-71), we see that this is equal to the mass times the absolute acceleration.

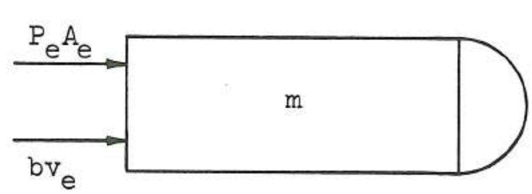

Figure 4-6

The conclusion to be emphasized at this point is that Newton's equation of motion

$$\overline{F} = m\,\overline{a}$$

applies instantaneously to a group of particles translating together, even though the total mass of the group may be changing due to a loss or gain of particles. Note that the term $\dot{m}\,\overline{v}$ does not appear in this equation, implying that the equation

$$\overline{F} = \frac{d}{dt}(m\,\overline{v})$$

does not apply to a system whose mass is changing, but rather it applies just to the case of systems with constant total mass (or when considering relativistic effects).

Note also that the acceleration of a changing system of particles is the acceleration of the center of mass of the system, assuming that the total mass is fixed for that instant. For example, in the case of solid-fuel rockets, it does not include the effect of center of mass motion relative to the rocket structure as the fuel is burned.

Control volume approach. Another approach to the derivation of the rocket acceleration equation is provided by a more general analysis of the material within a given control volume, as shown in Figure 4-7. This control volume may move or change its shape, and, in general, there is a

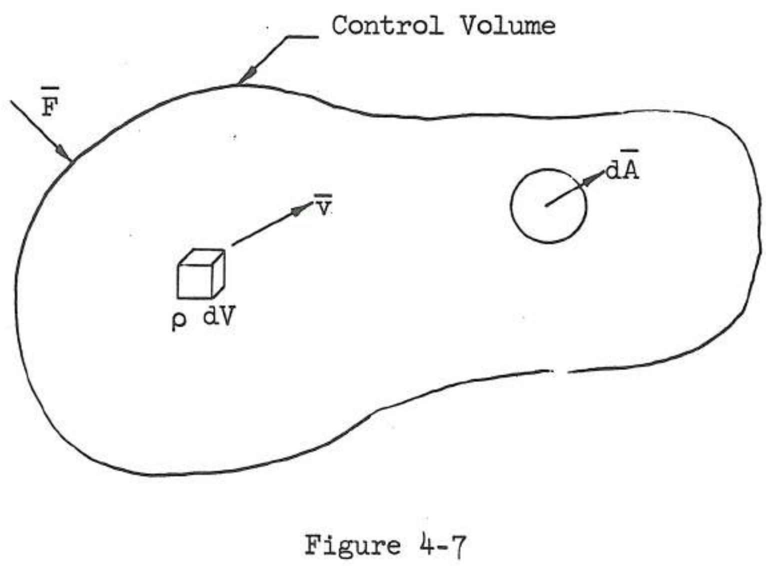

Figure 4-7

mass flow through its surface. The problem is to find the rate of change of linear momentum of a specific mass of fluid and solid material, i.e., of a certain set of particles, due to the resultant external force $\overline{F}$ acting on it.

Let us take as the system the material within the control volume at a given time t_1 and compare its momentum with the momentum of the same set of particles at time $t_1 + \Delta t$. The total linear momentum at time t_1 is just

$$\int_V \rho \, \overline{v} \, dV \Big|_{t_1}$$

where ρ is the mass density and $\overline{v}$ is the absolute velocity of the volume element dV. The integral is taken over the entire control volume V.

Now if we evaluate the same integral at time $t_1 + \Delta t$, we find that a slightly different set of particles is within the control volume. So, in order to follow the same set of particles, we must add to the momentum of the particles within the control volume at $t_1 + \Delta t$ the momentum of the "native" particles that left it in the interval Δt, and subtract the momentum of the "foreign" particles that entered it in the same interval. Therefore at time $t_1 + \Delta t$, the momentum of the original set of particles is

$$\int_V \rho \, \overline{v} \, dV \Big|_{t_1 + \Delta t} + \Delta t \int_A \rho \, \overline{v} \, (\overline{v}_r \cdot \overline{dA})$$

where the second integral accounts for the momentum of the particles entering or leaving the control volume in the interval Δt. This integral is over the entire surface of the control volume, $\overline{dA}$ being a surface element, the orientation being specified by the outward pointing normal vector. The vector $\overline{v}_r$ is the velocity, relative to the surface, of the particles that are entering or leaving. Thus $\rho(\overline{v}_r \cdot \overline{dA}) \, \Delta t$ represents a mass element crossing the surface in the interval Δt, with a positive $\overline{v}_r \cdot \overline{dA}$ referring to a leaving native element, and a negative $\overline{v}_r \cdot \overline{dA}$ referring to an entering foreign element. Multiplying each mass element crossing the boundary surface by its absolute velocity $\overline{v}$ and integrating over all all elements we obtain a correction term which can be interpreted as the net momentum outflow from the control volume during the interval Δt. Changes in $\overline{v}$ or $\overline{v}_r$ during this interval can be neglected.

Thus, the change in linear momentum of the original system of particles in the interval Δt is given by

$$\int_V \rho \, \bar{v} \, dV \Big|_{t_1 + \Delta t} - \int_V \rho \, \bar{v} \, dV \Big|_{t_1} + \Delta t \int \rho \, \bar{v}(\bar{v}_r \cdot \overline{dA})$$

Dividing the above expression by Δt and taking the limit as Δt approaches zero we obtain the rate of change of the total linear momentum of the system, and this, by Newton's equation, must be equal to the total external force on the system. Thus,

$$\bar{F} = \frac{\partial}{\partial t}\left[\int_V \rho \, \bar{v} \, dV \right] + \int_A \rho \, \bar{v}(\bar{v}_r \cdot \overline{dA}) \qquad (4\text{-}74)$$

Now let us apply Equation (4-74) to rocket previously considered in Figure 4-4. Take a control volume which includes the rocket and moves with it, as shown in Figure 4-8. The exit area of the rocket is a portion of the surface of the control volume.

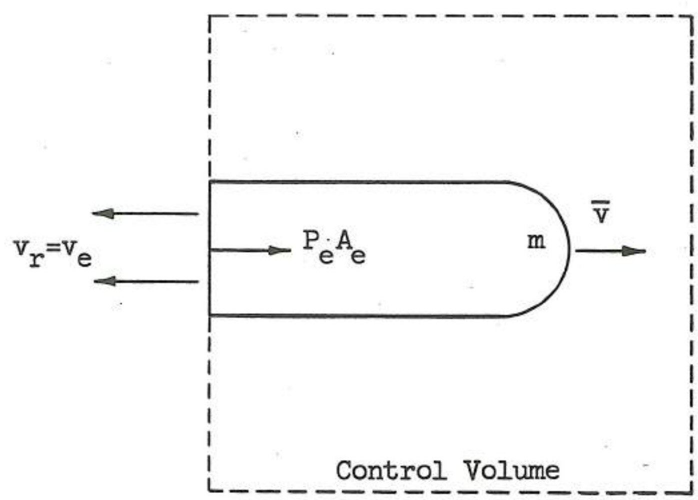

Figure 4-8

The total external force is due to the exit pressure p_e acting on the flat exit area A_e.

$$F = p_e \, A_e \qquad (4\text{-}75)$$

where the force acts to the right in the direction of the rocket velocity $\overline{v}$. Evaluating the magnitudes of the other terms of Equation (4-74), assuming positive to the right, we obtain

$$\left| \int_V \rho \, \overline{v} \, dV \right| = m \, v \tag{4-76}$$

resulting in

$$\left| \frac{\partial}{\partial t} \left[\int_V \rho \, \overline{v} \, dV \right] \right| = m \, \dot{v} + \dot{m} \, v = m \, \dot{v} - b \, v \tag{4-77}$$

where $\dot{m}$ is evaluated from Equation (4-67). The jet exhaust furnishes the only particles crossing the surface of the control volume, their relative velocity being v_e to the left, corresponding to an absolute velocity $v - v_e$ to the right. Therefore, using Equation (4-68), we find that

$$\left| \int_A \rho \, \overline{v} \, (\overline{v}_r \cdot \overline{dA}) \right| = \rho_e \, (v - v_e) \, v_e \, A_e = b \, (v - v_e) \tag{4-78}$$

Finally, from Equations (4-74), (4-75), (4-77), and (4-78) we obtain

$$p_e A_e = m \, \dot{v} - b \, v + b \, (v - v_e)$$

or

$$m \, \dot{v} = p_e A_e + b \, v_e$$

in agreement with Equation (4-71).

It can be seen that both this derivation and the previous one have neglected changes with time of the momentum of the jet as it flows within the control volume.

Of course, in practical cases, other external forces may act on the rocket such as aerodynamic and gravitational forces. Note that Equation (4-73) for the static thrust applies also for the case where the test is not made in a vacuum, providing that p_e is interpreted as the average gauge pressure at the exit.

Integration of the rocket equation. Let us return now to the differential equation for rocket flight in a vacuum with no gravitational forces, as given in Equations (4-71) and (4-72).

$$m \dot{v} = p_e A_e + b\ v_e = F_s$$

where the thrust force F_s is assumed to be constant. We can write, using Equation (4-67),

$$m \frac{dv}{dt} = m \frac{dv}{dm} \frac{dm}{dt} = - b\ m \frac{dv}{dm}$$

giving

$$\frac{dv}{dm} = - \frac{F_s}{bm} \tag{4-79}$$

Integrating between initial and final (normally burnout) conditions, designated by the subscripts i and f, respectively, we obtain

$$\int_{v_i}^{v_f} dv = - \frac{F_s}{b} \int_{m_i}^{m_f} \frac{dm}{m} \tag{4-80}$$

or

$$v_f - v_i = \frac{F_s}{b} \ln \frac{m_i}{m_f} \tag{4-81}$$

It can be seen in this case of no gravitational or aerodynamic forces that the velocity gain $v_f - v_i$ is independent of the burning time for a given mass ratio m_i/m_f so long as F_s/b is constant. The coefficient F_s/b has the units of impulse per unit mass. Normally it is specified in terms of the <u>specific impulse</u>

$$I_{sp} = \frac{F_s}{bg} \tag{4-82}$$

which is the total impulse per pound of propellant, the weight being measured at the earth's surface. So the velocity change written in terms of the specific impulse of the propellant is

$$v_f - v_i = I_{sp}\ g \ln \frac{m_i}{m_f} \tag{4-83}$$

For the case where the rocket is fired vertically upward in a constant gravitational field, the effect of the gravitational force $- mg$

can be superimposed upon the solution as given by Equation (4-83). This follows from the fact that Equation (4-71) is a linear differential equation and the superimposed gravitational force is not a function of position or velocity. Assuming a burning time T_b, the burnout velocity is

$$v_f = v_i - g\,T_b + I_{sp}\,g\,\ln\frac{m_i}{m_f} \qquad (4\text{-}84)$$

4-9. Examples

Example 4-1. Mass m_1, moving in the x direction with velocity v, hits m_2 and sticks to it (Figure 4-9). If all three particles are of equal mass m and if m_2 and m_3 are connected by a rigid, massless rod as shown, find the motion of the particles after impact. All particles can move without friction on the horizontal xy plane.

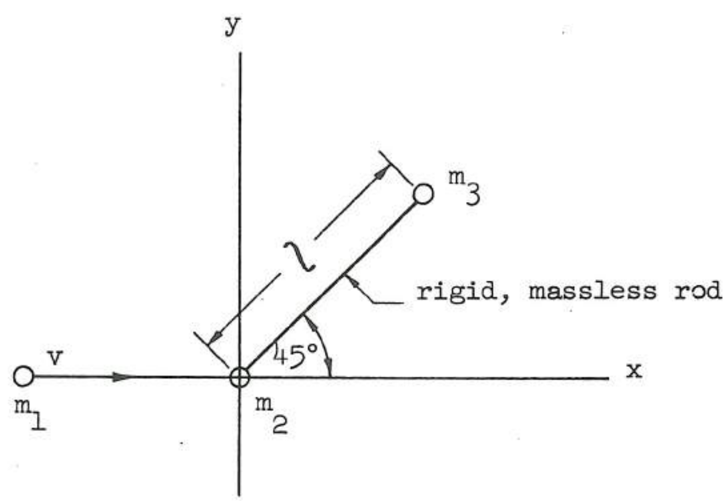

Figure 4-9

First we notice that the linear momentum of the system of three particles is conserved since no external forces act on the system. Let us take components of momentum along the rod. Since the rod is rigid, the velocity component along the rod after impact will be the same for all three masses. Calling this velocity u_a' we can write the equation of momentum conservation in this direction, obtaining

$$\frac{mv}{\sqrt{2}} = 3\,m\,u_a'$$

or

$$u_a' = \frac{1}{3\sqrt{2}}\,v$$

Similarly we can write the equation of momentum conservation in a direction perpendicular to the rod

$$m \frac{v}{\sqrt{2}} = 2\ m\ u'_p$$

or

$$u'_p = \frac{1}{2\ \sqrt{2}}\ v$$

where u'_p is the component of the velocity of m_1 and m_2 perpendicular to the rod immediately after impact. The velocity of m_3 just after impact is entirely in a direction along the rod since it receives an impulse in this direction through the rod. The above result can also be obtained by conserving angular momentum about a point fixed in the xy plane at the initial position of m_3.

The velocities of $(m_1 + m_2)$ and m_3 immediately after impact are as shown in Figure 4-10. Throughout the whole problem, i.e., before, during, and after impact, the center of mass of the system moves with a constant velocity $v/3$ in the positive x direction along a line

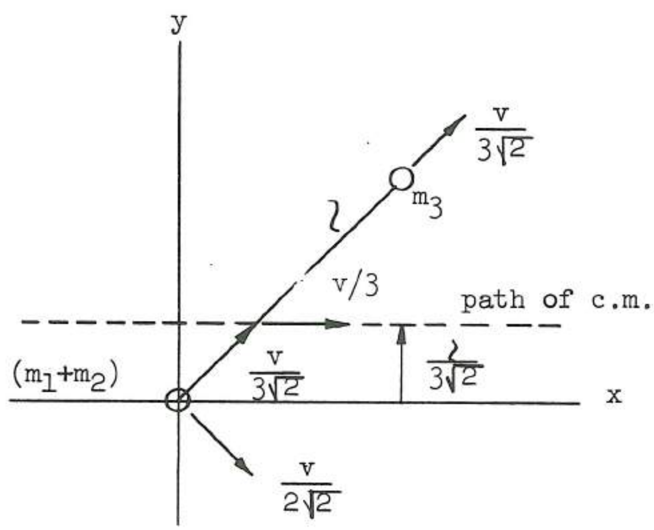

Figure 4-10

$y = l/3\ \sqrt{2}$. After impact the system rotates at a constant rate ω found by dividing the perpendicular velocity component of $(m_1 + m_2)$ relative to m_3 by the rod length l

$$\omega = \frac{v}{2\sqrt{2}\ l}$$

The angular velocity ω is constant after impact because no moments act on the system and, therefore, the angular momentum about the center of mass is conserved.

<u>Example 4-2</u>. Mass m_1 hits m_2 with inelastic impact $(e = 0)$ while moving with velocity v along the common line of centers of the three equal masses. Initially, masses m_2 and m_3 are stationary and the spring is unstressed. Find:

a. The velocities of m_1, m_2, and m_3 immediately after impact.
b. The maximum kinetic energy of m_3.
c. The minimum kinetic energy of m_2.
d. The maximum compression of the spring.
e. The final motion of m_1.

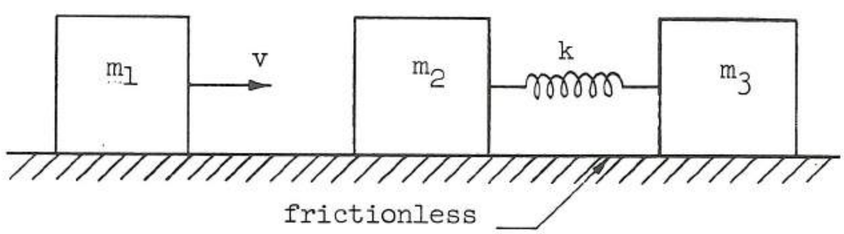

Figure 4-11

The problem is one-dimensional in nature and no external forces act on the system in this direction. Therefore, linear momentum is conserved throughout the problem. Because $e = 0$, masses m_1 and m_2 will move at the same velocity immediately after impact. Calling this velocity u_2' we can use the principle of conservation of linear momentum to obtain

$$m\ v = 2\ m\ u_2'$$

or

$$u_2' = \frac{1}{2}\ v$$

where m is the mass of each particle. Mass m_3 does not move at the time of impact because the spring is initially unstressed and $(m_1 + m_2)$ must move through a displacement before any force acts on m_3. So the situation immediately after impact is as shown in Figure 4-12.

For all time after the impact the total energy is conserved, and it is equal to the kinetic energy just after impact. Designating the displacements of $(m_1 + m_2)$ and m_3 by x_2 and x_3, respectively we can write

$$m\, v_2^2 + \frac{1}{2}\, m\, v_3^2 + \frac{1}{2}\, k(x_2 - x_3)^2 = \frac{1}{4}\, m\, v^2$$

where v_2 and v_3 are the velocities of $(m_1 + m_2)$ and m_3, respectively. From the conservation of linear momentum we obtain

$$2\, m\, v_2 + m\, v_3 = m\, v$$

from which we can solve for v_2 in terms of v and v_3, assuming, of course, that m_1 and m_2 continue to move together.

$$v_2 = \frac{1}{2}(v - v_3)$$

Because of total energy conservation, the total kinetic energy will be maximum when the potential energy stored in the spring is zero, i.e., when $x_2 = x_3$. So, setting $x_2 = x_3$ and substituting for v_2 in the energy equation given above, we obtain

$$\frac{1}{4}\,(v - v_3)^2 + \frac{1}{2}\, v_3^2 = \frac{1}{4}\, v^2$$

or

$$v_3(3\, v_3 - 2v) = 0$$

from which we find the roots

$$v_3 = 0,\ \frac{2}{3}\, v$$

Now the extreme values of kinetic energy for the particles (m_1+m_2) and m_3, taken separately, will occur when the individual velocities reach extreme values, i.e., when the accelerations and the spring force are zero and $x_2 = x_3$. Therefore, the extreme values of kinetic energy of the individual particles occur when the total kinetic energy is maximum, that is, when $v_3 = 0$ or $\frac{2}{3}\, v$. The maximum kinetic energy of m_3 is thus

$$(T_3)_{max} = \frac{1}{2}\, m(\frac{2}{3}\, v)^2 = \frac{2}{9}\, m\, v^2$$

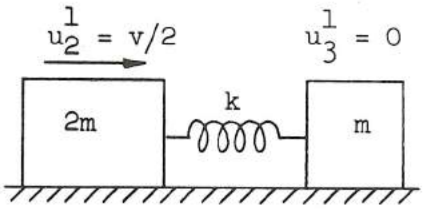

Figure 4-12

Because of linear momentum conservation, the minimum velocity of m_2 to the right occurs at the same time and is equal to

$$v_2 = \frac{1}{2m}\left[m\ v - \frac{2}{3}\ m\ v\right] = \frac{1}{6}\ v$$

Thus

$$(T_2)_{min} = \frac{1}{2}\ m\ (\frac{v}{6})^2 = \frac{1}{72}\ m\ v^2$$

The maximum compression of the spring occurs when the relative velocity of its two ends is zero, in which case all three particles are moving with the same velocity. By conservation of linear momentum

$$v_1 = v_2 = v_3 = \frac{1}{3}\ v$$

Substituting these values into the general energy equation we obtain the maximum spring compression

$$(x_2 - x_3)_{max} = \sqrt{\frac{m}{6k}}\ v$$

where the other sign of the square root is omitted because it implies spring tension, in which case the particles m_1 and m_2 have separated. As a matter of fact, as mass m_2 accelerates from its minimum velocity $\frac{1}{6}$ v, it leaves m_1 which continues to move at the uniform velocity $\frac{1}{6}$ v. Meanwhile, the center of mass of m_2 and m_3 translates uniformly at $\frac{5}{12}$ v in accordance with the conservation of total linear momentum, while m_2 and m_3 oscillate relative to it at an angular frequency $\omega = \sqrt{\frac{2k}{m}}$.

Before leaving this example we should notice that it could have been solved by considering the whole process from the viewpoint of an observer translating uniformly with the center of mass of the system. In this case the situation immediately after impact appears as shown in Figure 4-13. The center of mass appears fixed in this system and the given

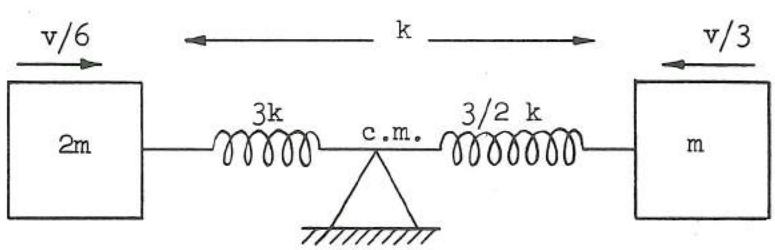

Figure 4-13

spring can be divided into two springs with stiffnesses that are inversely proportional to their lengths, as shown. Each mass executes simple harmonic motion at an angular frequency $\omega = \sqrt{\frac{3k}{2m}}$. This motion continues only for a half-cycle at which time m_1 and m_2 separate and the motion continues in the manner explained previously.

Example 4-3. A two-stage sounding rocket is to be shot vertically from the earth. Each stage individually has a mass ratio (initial mass to final mass) equal to 9, the total weight of the first stage being 360 lbs. and that of the second stage being 90 lbs. Assuming that a payload weight of 30 pounds is mounted atop the second stage and that a 20-second coasting period occurs between first stage burnout and the ignition of the second stage, find the maximum altitude reached by the second stage plus payload. The burning time is assumed to be short and the gravity change is neglected during the coasting phase before second stage ignition. The specific impulse of the propellant is $I_{sp} = 250$ sec. Assume a nonrotating earth and neglect atmospheric drag.

It can be seen that the effective mass ratio for each stage is the same, namely,

$$\frac{480}{160} = \frac{120}{40} = 3$$

Using Equation (4-83) we find that the velocity gained per stage is

$$\Delta v = (250)(32.2) \ln 3 = 8850 \text{ ft/sec.}$$

We find, using Equations (3-6) and (3-7), that the velocity and altitude after the 20-second coasting period is

$$v = 8850 - (32.2)(20) = 8206 \text{ ft/sec.}$$

$$h = (8850)(20) - \frac{1}{2}(32.2)(400) = 170{,}600 \text{ ft.}$$

The impulse of the second stage increases this velocity to 8206 + 8850 = 17,056 ft/sec. The maximum altitude is found from the conservation of energy. At second stage burnout the total energy per unit mass is found, using Equations (3-37) and (3-60), to be

$$\frac{1}{2}(1.706)^2 \times 10^8 - (32.2)(3960)(5280)\left[1 + \frac{1.706 \times 10^4}{(3960)(5280)}\right]^{-1} = -5.273 \times 10^8 \text{ ft}^2/\text{sec}^2$$

Using Equation (3-60) again, we can solve for the maximum altitude

$$h_{max} = (3960)(5280)\left[\frac{(32.2)(3960)(5280)}{5.273 \times 10^8} - 1\right] = 5.790 \times 10^6 \text{ ft} \approx 1100 \text{ mi.}$$

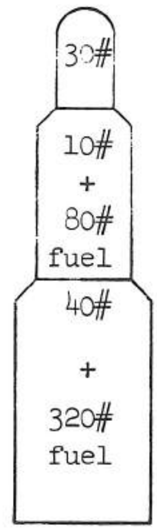

Figure 4-14

It is interesting to note that if a constant acceleration of gravity g = 32.2 ft/sec^2 had been used throughout, the maximum height would have been calculated to be only about 890 mi. So the variation of gravitational force with altitude has a significant effect in this example.

Note also that the coasting before second stage ignition resulted in a delay of 20 sec in obtaining the velocity increase due to this second stage, i.e., it cost a velocity decrease of 8850 ft/sec for 20 sec. Thus the maximum altitude was reduced by 1.77×10^5 ft or 33.5 mi. In the practical case, of course, atmospheric drag forces which are strongly dependent upon velocity would occur. Thus, the coasting to a higher altitude with its lower atmospheric density before second stage ignition would actually result in smaller losses than if the stages were fired in quick succession.

Chapter 5

ORBITAL MOTION

This chapter will be concerned with the calculation of the path in space followed by a particle as it moves in a gravitational field. Two general types of gravitational fields will be considered: (1) a uniform field, and (2) an inverse-square attraction toward a point. Most of the emphasis will be on the latter, but the former is included as a relatively simple but important special case.

In the context of vehicle dynamics, we will be solving for the translational motion due to gravity of the center of mass, the only restriction being that the external field be essentially uniform in the region occupied by the vehicle at any given instant. This follows directly from Eqn. (4-6) and implies that the translational motion is as though the entire mass were concentrated at the mass center.

5-1. Motion of a Particle in a Uniform Gravitational Field

In Sec. 3-1 under Case 1 we solved for the one-dimensional motion of a uniformly accelerated particle. The more general motion includes also a constant velocity component in a direction normal to the uniform force field, thereby confining the motion to a plane.

For the case of a particle moving under the influence of a uniform gravitational field, let us choose the xy plane as the plane of motion with the y-axis directed vertically upward. Assume for convenience that the particle is located at the origin O at time $t = 0$ and is moving with velocity v_o at an angle γ above the horizontal, as shown in Fig. 5-1.

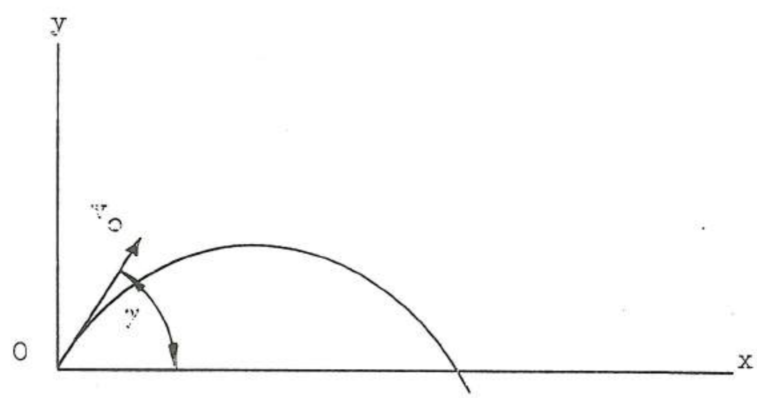

Fig. 5-1

The initial values of the velocity components are

$$\dot{x}(0) = v_o \cos \gamma$$
$$\dot{y}(0) = v_o \sin \gamma \qquad (5\text{-}1)$$

Noting that the acceleration components are

$$a_x = 0$$
$$a_y = -g \qquad (5\text{-}2)$$

we obtain the displacement components from Eqn. (3-7), namely,

$$x = v_o t \cos \gamma \qquad (5\text{-}3)$$

$$y = v_o t \sin \gamma - \frac{1}{2} g t^2 \qquad (5\text{-}4)$$

Eliminating t between Eqns. (5-3) and (5-4) we obtain the trajectory

$$y = x \tan \gamma - \frac{g}{2 v_o^2 \cos^2 \gamma} x^2 \qquad (5\text{-}5)$$

which is the equation of a vertical parabola. The location of the vertex (x_v, y_v) is also the point of zero slope in this case and is found by differentiating Eqn. (5-5) with respect to x, resulting in

$$x_v = \frac{v_o^2}{2g} \sin 2\gamma \qquad (5\text{-}6)$$

$$y_v = \frac{v_o^2}{2g} \sin^2 \gamma \qquad (5\text{-}7)$$

In general, the time required to go to a given value of x is

$$t = \frac{x}{\dot{x}(0)} = \frac{x}{v_o \cos \gamma} \qquad (5\text{-}8)$$

since the x-component of the velocity is constant. Therefore, the time required to reach the vertex is

$$t_v = \frac{v_o}{g} \sin \gamma \tag{5-9}$$

This value is obtained also from Eqn. (3-6) by setting $v_y = 0$.

For the case of a trajectory over a flat surface, the range is

$$x = 2x_v = \frac{v_o^2}{g} \sin 2\gamma \tag{5-10}$$

and the time of flight is

$$t = 2t_v = \frac{2v_o}{g} \sin \gamma \tag{5-11}$$

It can be seen that the maximum range is achieved for $\gamma = 45°$, and the maximum time of flight for $\gamma = 90°$.

Now let us calculate the initial flight path angle γ such that the particle passes through a given point P at (x,y), as shown in Fig. 5-2. In general, two

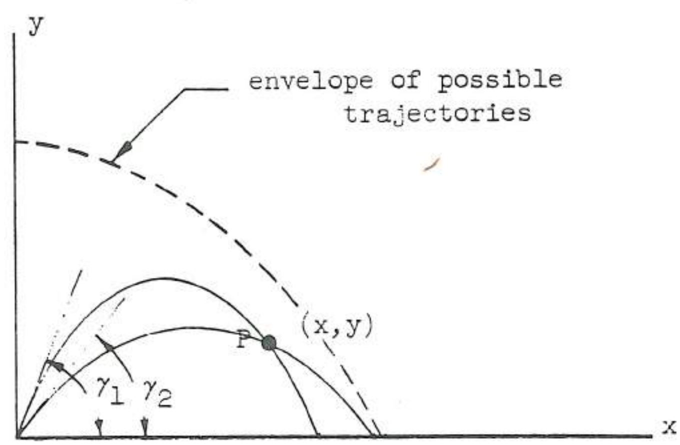

Fig. 5-2

values of γ are possible. These values γ_1 and γ_2 are the roots of the quadratic equation in tan γ obtained from Eqn. (5-5).

$$\frac{gx^2}{2v_o^2}(1 + \tan^2\gamma) - x \tan \gamma + y = 0$$

or

$$\tan^2\gamma - \frac{2v_o^2}{gx}\tan\gamma + \frac{2v_o^2}{gx^2}y + 1 = 0 \qquad (5\text{-}12)$$

where, of course, the roots must be real in order to assure the existence of a trajectory through the point P for a given v_o.

The maximum range in a given direction corresponds to a double root. The value of this root is found from Eqn. (5-12) by completing the square, i.e.,

$$\left(\tan\gamma - \frac{v_o^2}{gx}\right)^2 = \tan^2\gamma - \frac{2v_o^2}{gx}\tan\gamma + \frac{2v_o^2}{gx^2}y + 1 = 0 \qquad (5\text{-}13)$$

from which we obtain that

$$\tan\gamma = \frac{v_o^2}{gx_e} \qquad (5\text{-}14)$$

where γ is the initial flight path angle such that the trajectory just reaches a point (x_e, y_e) on the envelope of possible trajectories for a given v_o. From Eqn. (5-13) we can obtain the equation of the envelope:

$$\left(\frac{v_o^2}{gx_e}\right)^2 = \frac{2v_o^2}{gx_e^2}y_e + 1$$

or

$$x_e^2 = -\frac{2v_o^2}{g}\left(y_e - \frac{v_o^2}{2g}\right) \qquad (5\text{-}15)$$

which is parabolic in form with the vertex located at $x = 0$, $y = v_o^2/2g$.

Assuming that the azimuth angle (i.e., the direction of the horizontal velocity component) of the trajectory is arbitrary, any point within the paraboloid formed by rotating the envelope about the vertical line $x = 0$ will have at least one trajectory passing through it and will thus be within the range of a projectile of initial velocity v_o.

Example 5-1. Assuming a given v_o, find the flight path angle such that a maximum range is achieved in a direction 45° above the horizontal.

In this case the trajectory passes through a point $x_e = y_e$ on the trajectory envelope, since the maximum range is to be calculated at 45° above the horizon. Dividing Eqn. (5-15) by x_e^2, we obtain

$$1 = -\frac{2v_0^2}{gx_e}\left(\frac{y_e}{x_e} - \frac{v_0^2}{2gx_e}\right)$$

or, using Eqn. (5-14),

$$-2 \tan \gamma \left(1 - \frac{1}{2} \tan \gamma\right) = 1$$

from which we obtain

$$\tan \gamma = 1 + \sqrt{2}$$

or

$$\gamma = 67\tfrac{1}{2}\,^\circ$$

The range is found from Eqn. (5-14) and is equal to

$$\sqrt{2}x_e = \frac{v_0^2}{g}\left(\frac{\sqrt{2}}{1 + \sqrt{2}}\right)$$

5-2. Kepler's Laws and Newton's Law of Gravitation

We turn now to the study of the motion of a particle under the influence of an inverse-square gravitational attraction.

The principal factors that influence the gravitational force on a body were originally deduced by studying planetary motions. Kepler, after a careful analysis of the observational data of Tycho Brahe, found that he could predict the motions of the planets on the basis of the following assumptions:

1. The orbit of each planet is an ellipse with the sun at one focus.

2. The radius vector drawn from the sun to a planet sweeps over equal areas in equal times.

3. The squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.

These are Kepler's laws. The first two laws were published in 1609 and the third in 1619. Thus, they preceded Newton's laws of motion by nearly 70 years. As a matter of fact, Newton deduced from Kepler's laws and his own laws of motion that the gravitational force between two particles lies along the straight line connecting them and is of the form

$$F_r = -G\frac{m_1 m_2}{r^2} \tag{5-16}$$

where m_1 and m_2 are the masses of the particles, r is the distance between them, and G is a universal constant that is independent of the nature of the masses or their location in space. The minus sign signifies a force of attraction; the value of the universal gravitational constant is

$$G = 6.67 \times 10^{-8}\,\text{cm}^3/\text{gram sec}^2 = 3.44 \times 10^{-8}\,\text{ft}^4/\text{lb sec}^4$$

Eqn. (5-16) is a statement of Newton's law of gravitation.

In addition, Newton showed that the gravitational force of a uniform spherical shell on an external particle is the same as if the mass of the shell were concentrated at its center. By superimposing the effects of successive spherical layers, it is then apparent that the masses of the sun and planets can be considered to be concentrated at their respective centers, providing that the density of each is spherically symmetric about its center.

5-3. The Two-Body Problem

Let us consider the mutual gravitational attraction between two spherical masses, which, as we have seen, can be considered as particles in calculating

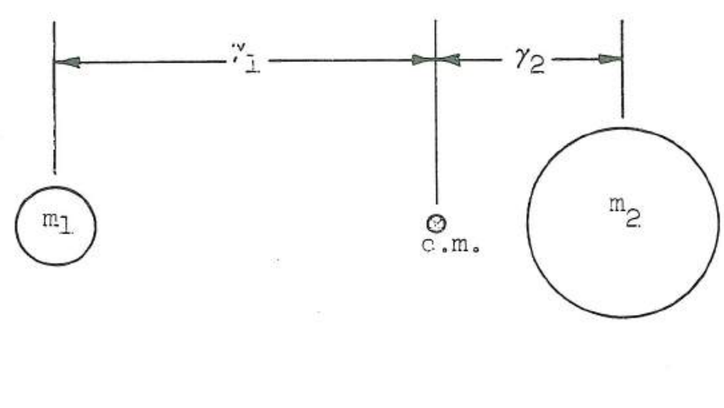

Fig. 5-3

their translational motion. In accordance with Newton's law of gravitation the force is

$$f_{12} = f_{21} = -G \frac{m_1 m_2}{(r_1 + r_2)^2} \tag{5-17}$$

Because r_1 and r_2 are measured from the center of mass, we know that

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \tag{5-18}$$

Therefore,

$$f_{12} = -\frac{K_1}{r_1^2} \tag{5-19}$$

where

$$K_1 = G \frac{m_1 m_2}{\left(1 + \frac{m_1}{m_2}\right)^2} \tag{5-20}$$

and, similarly,

$$f_{21} = -\frac{K_2}{r_2^2} \tag{5-21}$$

where

$$K_2 = G \frac{m_1 m_2}{\left(1 + \frac{m_2}{m_1}\right)^2} \tag{5-22}$$

The general motion of the system of two particles will involve a uniform translation of the center of mass, assuming that no external forces act on the system. Therefore, we can always choose an inertial reference frame in which the center of mass is fixed. With respect to this reference frame, we see from Eqns. (5-19) and (5-21) that each mass moves as though it is attracted by an inverse-square force to the fixed center of mass. Thus, the two-body problem reduces to the motion of individual particles attracted by an inverse-square force to a fixed point.

Another approach that is commonly used in astronomical work is to consider the larger mass (say m_2) as fixed and adjust the value of K_1, such that the proper relative motion occurs. Calling this adjusted value K_1', we find that

$$K_1' = K_1 \left(\frac{r_1 + r_2}{r_1}\right)^3 = Gm_1 (m_1 + m_2) \qquad (5\text{-}23)$$

The correction factor is seen to be $(r/r_1)^3$. The factor r^2/r_1^2 occurs because of the inverse-square nature of the force, while the remaining r/r_1 factor is introduced to account for the linear increase of inertial forces with r, assuming that the angular rate of the radius vector is unchanged.

Thus, the motion of m_1 relative to m_2 can be found by assuming that

$$f_{12} = -\frac{K_1'}{r^2} = -G\,\frac{m_1(m_1 + m_2)}{r^2} \qquad (5\text{-}24)$$

It is to be emphasized that f_{12} as given by Eqn. (5-24) is not the actual force on m_1, but rather is that force which gives the proper relative motion, assuming m_2 is fixed.

Also, if one assumes m_2 is fixed and uses Eqn. (5-24) as the force equation, then any calculations of the total kinetic or potential energy of the system will be incorrect unless one used the reduced mass

$$m_1' = \frac{m_1}{1 + \frac{m_1}{m_2}} \qquad (5\text{-}25)$$

in place of the actual mass m_1. It can be seen that if $m_2 \gg m_1$, then the reduced mass is approximately equal to the actual mass.

We can summarize by saying that the two-body problem can be reduced to the problem of solving for the motion of a single particle under the influence of an inverse-square attraction to a given reference point, the point being either (1) the other particle, or (2) the center of mass of the system. Having simplified the statement of the problem, we will solve for the orbit in Sec. 5-4.

5-4. Determination of the Orbit

Let us consider again the system of Fig. 5-3 and write the equations of motion for m_1. First, notice that the motions of both m_1 and m_2 are confined to a single plane, namely, the plane determined by the relative velocity vector

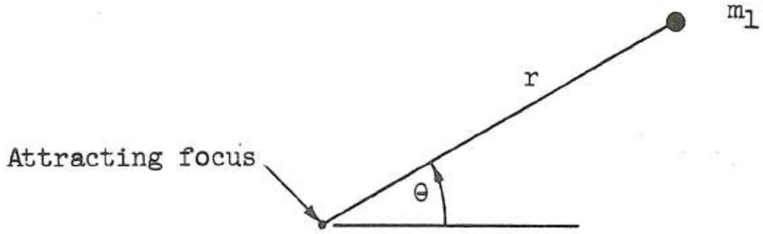

Fig. 5-4

and the radial line joining the particles. This follows from the fact that the forces on m_1 and m_2 have no components perpendicular to this plane. So we can define the position of m_1 relative to the attracting center or focus in terms of the usual polar coordinates r and Θ. Taking r and Θ components of force and acceleration, we can write, using Eqns. (2-21) and (3-1), that

$$F_r = m_1 a_r = m_1(\ddot{r} - r\dot{\Theta}^2) \qquad (5\text{-}26)$$

$$F_\Theta = m_1 a_\Theta = m_1(r\ddot{\Theta} + 2\dot{r}\dot{\Theta}) \qquad (5\text{-}27)$$

But, from Eqns. (5-19) and (5-20), we see that

$$\frac{F_r}{m_1} = -\frac{\mu_1}{r_1^2} \qquad (5\text{-}28)$$

where

$$\mu_1 = G\frac{m_2}{\left(1 + \frac{m_1}{m_2}\right)^2} \qquad (5\text{-}29)$$

and the attracting focus is the center of mass. Alternatively, from Eqn. (5-24), we obtain

$$\frac{F_r}{m_1} = -\frac{\mu'}{r^2} \qquad (5\text{-}30)$$

where

$$\mu' = G(m_1 + m_2) \tag{5-31}$$

and the attracting focus is at m_2.

In any event we see that the equation for the radial acceleration can be put in the form

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \tag{5-32}$$

where, depending upon the choice of reference point, the constant μ is defined as the μ_1 of Eqn. (5-29) or as the μ' of Eqn. (5-31). It is generally most convenient to think of Eqn. (5-32) as a relation between the absolute acceleration in the radial direction and the force per unit mass toward a fixed attracting focus.

Noting that $F_\theta = 0$ we can write Eqn. (5-27) in the form

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{5-33}$$

Eqns. (5-32) and (5-33) constitute the differential equations of motion for the particle m_1.

Eqn. (5-33) can be integrated directly, giving

$$r^2\dot{\theta} = h \tag{5-34}$$

where the constant, h, is the angular momentum per unit mass, as can be seen from Eqn. (3-79). At this point, it is interesting to note that the areal velocity, i.e., the area swept over per unit time by the radius vector, is given by

$$\dot{A} = \frac{1}{2} r^2\dot{\theta} = \frac{1}{2} h \tag{5-35}$$

and is constant in accordance with Kepler's second law. Physically, it means that the angular momentum about the attracting focus is conserved.

Now let us make the substitution

$$u = \frac{1}{r} \tag{5-36}$$

and note from Eqn. (5-34) that

$$\dot{\theta} = hu^2 \tag{5-37}$$

Also, we obtain from Eqns. (5-36) and (5-37) that

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta} \tag{5-38}$$

and

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{5-39}$$

From Eqns. (5-32), (5-37), and (5-39) we find that

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\mu u^2$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{5-40}$$

The general solution of Eqn. (5-40) is mathematically similar to that obtained previously in Example 3-2 and is of the form

$$u = \frac{\mu}{h^2} + C\cos(\theta - \theta_o) \tag{5-41}$$

where C and θ_o are constants to be determined. The constant C can be evaluated in terms of h, the angular momentum per unit mass, and e, the total energy per unit mass. Note that h and e are independent of the particle position in a given orbit. They are known as the dynamical constants of the orbit.

Suppose that we set $\theta = \theta_o$. Then, from Eqns. (5-38) and (5-41) we see that $\dot{r} = 0$ at this time. Therefore, the total energy per unit mass is

$$e = \frac{1}{2}r^2\dot{\theta}^2 - \frac{\mu}{r} = \frac{1}{2}h^2u^2 - \mu u \tag{5-42}$$

where

$$u = \frac{\mu}{h^2} + C \qquad (5\text{-}43)$$

as can be seen from Eqn. (5-41). Finally, from Eqns. (5-42) and (5-43) we obtain

$$C^2 = \frac{\mu^2}{h^4} + \frac{2e}{h^2} \qquad (5\text{-}44)$$

By a proper choice of the reference line from which θ is measured, we can cause θ_o to be zero. Then, from Eqns. (5-41) and (5-44), and setting $\theta_o = 0$, we obtain the equation of the orbit:

$$r = \frac{\left(\frac{h^2}{\mu}\right)}{1 + \sqrt{1 + 2\,\frac{eh^2}{\mu^2}}\cos\theta} \qquad (5\text{-}45)$$

where we have arbitrarily chosen the positive square root for C and the reference line for θ is fixed accordingly.

So we see that, for a given gravitational coefficient μ, the size and shape of the orbit is entirely determined by the two <u>dynamical</u> <u>constants</u> e and h. Eqn. (5-45) is the polar form of the general equation of a conic section as shown in Sec. 5-5.

5-5. Geometry of Conic Sections

In this section we shall present the geometrical characteristics of conic sections that are most important for our purposes. With this background we will then be able to correlate the dynamical and geometrical characteristics of the various possible orbits.

The equation of the general conic, written in terms of polar coordinates, is

$$r = \frac{\ell}{1 + \epsilon \cos\theta} \qquad (5\text{-}46)$$

where ℓ is the semilatus rectum and ϵ is the eccentricity. It can be seen that ℓ is just the value of r corresponding to $\theta = \pm\,\pi/2$. Also, it is the parameter governing the <u>size</u> of the conic section. On the other hand, the eccentricity ϵ determines its <u>shape</u>, as we shall see.

<u>Ellipse</u>. The ellipse is a conic section for which $0 \leqslant \epsilon < 1$. (Consider the circle to be the special case for which $\epsilon = 0$.) It can be defined as the locus of points whose distance from a given point F (focus) is a constant factor ϵ times its distance from a straight line known as the directrix (Fig. 5-5). It is also the locus of points such that the sum of the distances to two foci, F and F', is a constant length 2a which is also the length of the

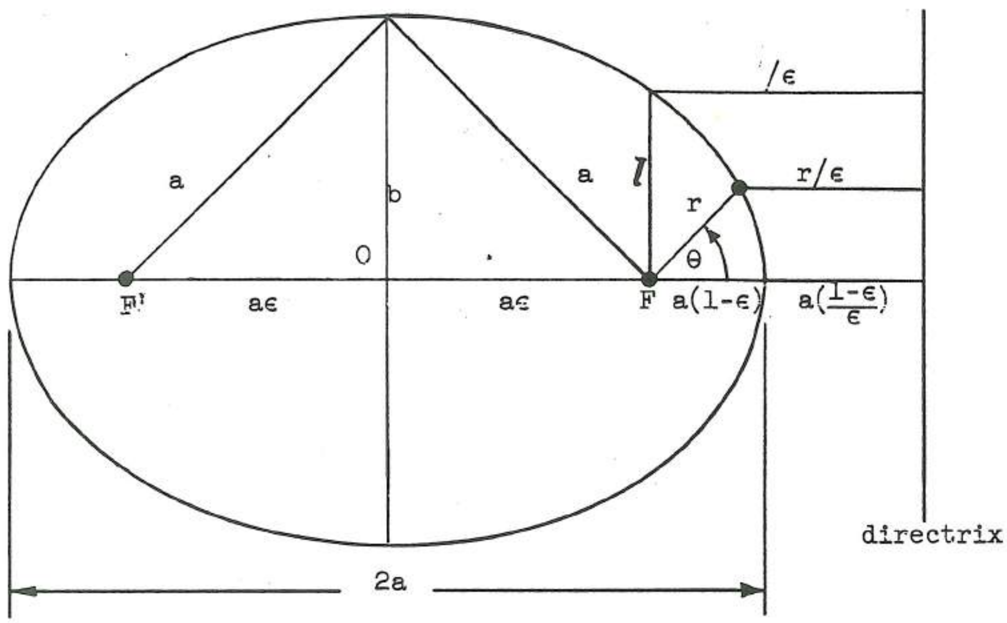

Fig. 5-5. Ellipse

major axis. The distance between the foci is just ϵ times the major axis. It can be seen from the figure that the semiminor axis b is related to the semimajor axis a by the equation

$$b = a\sqrt{1 - \epsilon^2} \tag{5-47}$$

Also, the semilatus rectum l is related to a by the equation

$$l = a(1 - \epsilon^2) \tag{5-48}$$

So, from Eqns. (5-46) and (5-48) one can write the equation of an ellipse in the form

$$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos \Theta} \tag{5-49}$$

This equation is convenient when the ellipse is given in terms of its geometrical constants a and ϵ.

It can be seen that the angle Θ is measured from the position where r is a minimum. For the case of orbits about the earth, this minimum r is known as the perigee distance

$$r_p = \frac{\ell}{1 + \epsilon} = a(1 - \epsilon) \tag{5-50}$$

and the corresponding point on the orbit is the perigee. Similarly, the point of maximum r is known as the apogee and the apogee distance is

$$r_a = \frac{\ell}{1 - \epsilon} = a(1 + \epsilon) \tag{5-51}$$

From Eqns. (5-50) and (5-51) we see that

$$a = \frac{1}{2}(r_p + r_a) \tag{5-52}$$

indicating the reason why a is sometimes known as the mean distance. (Note that a time average is not implied here.) From Eqns. (5-50) and (5-51) we also find that

$$\epsilon = \frac{1}{2a}(r_a - r_p) \tag{5-53}$$

The area of the ellipse can be calculated by noting that it is just the projected area of a circle of radius a onto a nonparallel plane, resulting in a foreshortening ratio b/a along the minor axis. Thus, the total area of the ellipse is

$$A = \pi a^2 \left(\frac{b}{a}\right) = \pi ab \tag{5-54}$$

Parabola. For a given perigee distance r_p, we see from Eqn. (5-50) that, as the eccentricity ϵ approaches unity, the semimajor axis a approaches infinity. Also, the second focus F' recedes toward infinity. In the limit, $\epsilon = 1$ and the conic section is parabolic in form, as shown in Fig. 5-6.

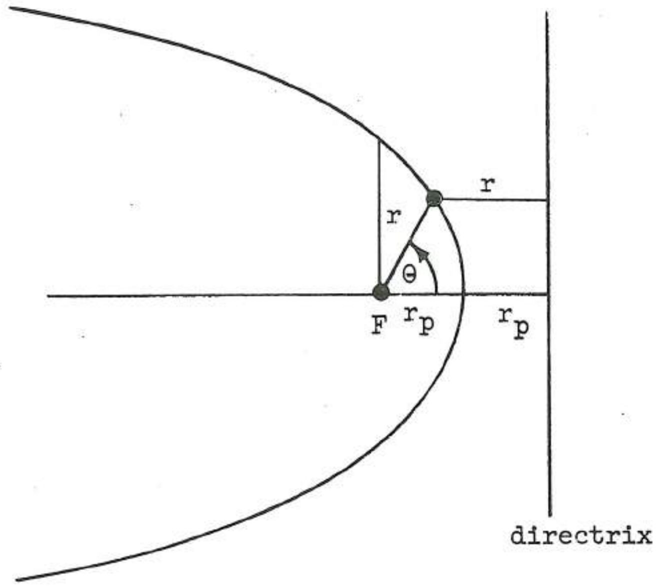

Fig. 5-6. Parabola

From Eqn. (5-46) we obtain the equation of the parabola:

$$r = \frac{l}{1 + \cos\Theta} \tag{5-55}$$

The semilatus rectum in this case is seen to be

$$l = 2r_p \tag{5-56}$$

<u>Hyperbola</u>. As we have seen, Eqn. (5-46) is valid for all conic sections and the resulting curve is a hyperbola for $\epsilon > 1$. The principal geometrical parameters are shown in Fig. 5-7. The hyperbola can be defined as the locus of points such that the difference of their distances from two fixed foci F and F' is a constant length 2a, where, as in the case of an ellipse, the separation of the foci is $2a\epsilon$. It can be seen that the hyperbola has two parts or <u>branches</u> which are separated by a distance 2a, again known as the major axis. These branches approach asymptotes making an angle ϕ with the major axis where

$$\phi = \cos^{-1}(1/\epsilon) \tag{5-57}$$

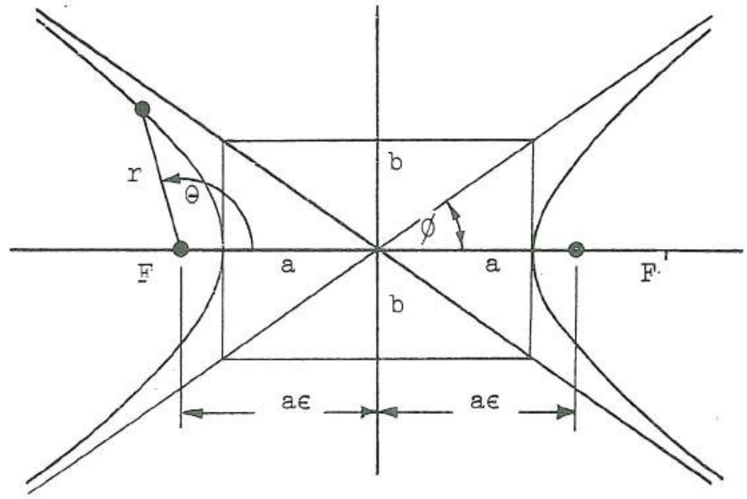

Fig. 5-7. Hyperbola

The semiminor axis b is the distance from the vertex to an asymptote, measured in a direction perpendicular to the major axis.

$$b = a\sqrt{\epsilon^2 - 1} \qquad (5\text{-}58)$$

Note that b need not be smaller than a. The semilatus rectum is

$$l = a(\epsilon^2 - 1) \qquad (5\text{-}59)$$

which can be substituted into Eqn. (5-46) to yield the general equation of the hyperbola:

$$r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos \theta} \qquad (5\text{-}60)$$

The perigee distance is

$$r_p = \frac{l}{1 + \epsilon} = a(\epsilon - 1) \qquad (5\text{-}61)$$

where the distance is measured from the attracting focus to the point of closest approach of the nearest branch. (The other branch gives the orbit for inverse-square repulsion by the same focus.)

5-6. Orbital Relationships

<u>Relation of dynamical and geometrical constants</u>. We have seen in Sec. 5-4 that the inverse-square attraction of a particle toward a fixed point results in an orbit that has the form of a conic section. In Sec. 5-5 we stated some of the geometrical characteristics of conic sections. Now let us see how the geometrical aspects of the orbit depend upon the dynamical constants of the motion.

First, we see by comparing Eqns. (5-45) and (5-46) that

$$\epsilon = \sqrt{1 + 2\,\frac{eh^2}{\mu^2}} \qquad (5\text{-}62)$$

and

$$\ell = \frac{h^2}{\mu} \qquad (5\text{-}63)$$

From Eqns. (5-48), (5-59), and (5-63) we obtain

$$a = \pm\,\frac{h^2}{\mu(1 - \epsilon^2)} \qquad (5\text{-}64)$$

or, substituting for ϵ from Eqn. (5-62),

$$a = \mp\,\frac{\mu}{2e} \qquad (5\text{-}65)$$

where the choice of signs refers to the elliptical and hyperbolic cases, respectively. This sign convention will be continued throughout this chapter, namely, the top sign refers to an elliptical orbit and the bottom sign to a hyperbolic orbit.

In general the total energy per unit mass is

$$e = \frac{1}{2}\,v^2 - \frac{\mu}{r} \qquad (5\text{-}66)$$

where v is the particle speed in the orbit. Note from Eqn. (5-65) that the total energy is negative for an elliptical orbit and positive for a hyperbolic orbit. It is zero for a parabolic orbit. Also, note that, for a given μ, the value of a is directly determined by the total energy e per unit mass.

From Eqns. (5-65) and (5-66) we can obtain an expression relating the speed to the distance from the focal point for a given orbit.

$$v^2 = \mu \left(\frac{2}{r} \mp \frac{1}{a}\right) \tag{5-67}$$

This is twice the kinetic energy per unit mass.

Orbital period. The period of the motion in an elliptical orbit is found by dividing the total area by the areal velocity. From Eqns. (5-35), (5-47), (5-48), (5-54), and (5-63), we see that the period T is given by

$$T = \frac{\pi ab}{\dot{A}} = 2\pi \sqrt{\frac{a^3}{\mu}} \tag{5-68}$$

This is a statement of Kepler's third law. Thus, we have seen that Kepler's laws apply exactly for the case of a particle under the influence of an inverse-square attraction to a fixed point. In the context of their original statement, i.e., the solar system, they are only approximately true because: (1) other planets cause orbital perturbations, and (2) the constant μ is not independent of the planet's mass. Other applications of Kepler's laws, such as their use in calculating satellite orbits about a planet, may show significant deviations from observations due to the fact that the planet is not exactly spherical but may be slightly oblate.

Eccentric anomaly. One is often faced with the problem of establishing the position of an orbiting body within its orbital plane at some future time. For a given gravitational constant μ, one needs to know two independent geometrical constants such as a and ϵ and also the time t since perigee. The calculations are aided by introducing the parameter E, known as the eccentric anomaly. Its geometric significance is shown in Fig. 5-8.

Recall that the ellipse can be considered as the projection of a circle of radius a on a non-parallel plane, causing a foreshortening factor b/a in the direction of the minor axis. Reversing this process, we find that the point P on the ellipse corresponds to the P' on the auxiliary circle. The angle E, measured at the center O, between the perigee and P' is known as the eccentric anomaly. The familiar angle θ giving the actual position P relative to the focus F is known as the true anomaly. The equation relating E and θ is found from Fig. 5-8 by noting that

$$AP = \frac{b}{a} AP'$$

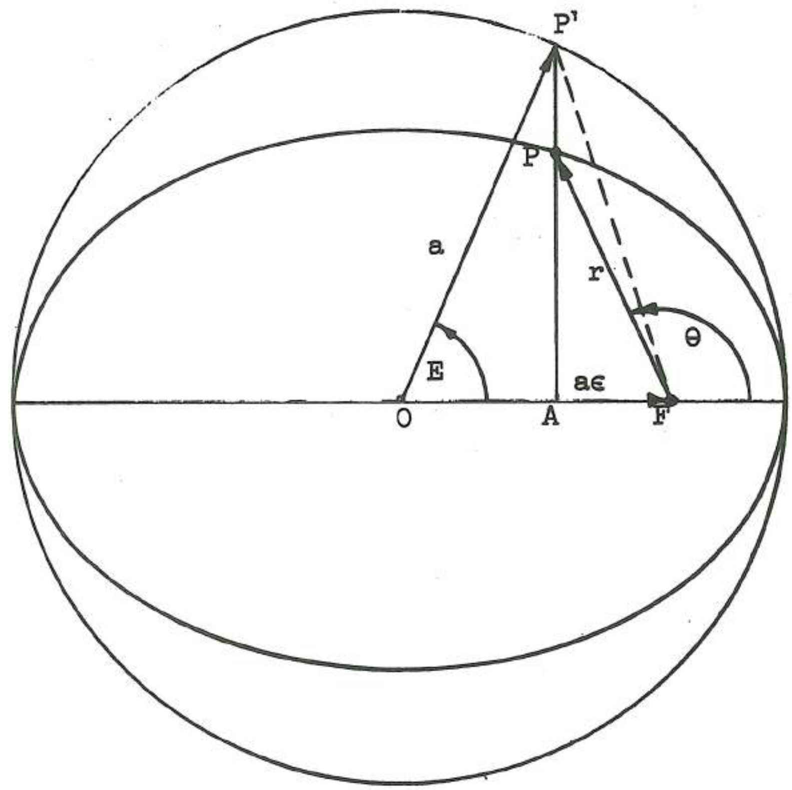

Fig. 5-8

where b/a is the foreshortening factor. Thus

$$r \sin \theta = \frac{b}{a} (a \sin E)$$

or using Eqns. (5-47) and (5-49), we obtain

$$\sin E = \frac{r}{b} \sin \theta = \frac{\sqrt{1 - \epsilon^2} \sin \theta}{1 + \epsilon \cos \theta} \tag{5-69}$$

which can be reduced by direct trigonometric substitution to the form

$$\tan \frac{E}{2} = \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \tan \frac{\theta}{2} \tag{5-70}$$

5-7. Time of Flight

Elliptical orbit. There remains the problem of relating the eccentric anomaly E to the time t since perigee passage. To do this we note that the radius vector FP sweeps out equal areas in equal times. Similarly, the line FP' sweeps over equal areas in equal times. Thus, the area swept over by FP' since perigee passage is just $(\pi a^2)t/T$. But this area is also the sector area $(1/2)a^2E$ minus the area of the triangle OFP'. So we can write

$$\frac{\pi a^2 t}{T} = \frac{1}{2} a^2 E - \frac{1}{2} a^2 \epsilon \sin E$$

or

$$t = \frac{T}{2\pi}(E - \epsilon \sin E) \qquad (5\text{-}71)$$

The quantity

$$M = E - \epsilon \sin E \qquad (5\text{-}72)$$

is called the mean anomaly and is the angular displacement of a line moving at the average angular rate $(2\pi/T)$ of the orbiting body.

In order to calculate the time of flight between any two points θ_1 and θ_2 in an elliptical orbit, we have merely to calculate $(t_2 - t_1)$ where t_1 and t_2 are the corresponding times since perigee. These are computed by first evaluating the eccentric anomaly E from Eqn. (5-69) or (5-70) and then using Eqn. (5-71) to obtain the time.

The reverse problem, namely, that of finding the position if the time since perigee is given, is more difficult. It involves solving Eqn. (5-72) for the eccentric anomaly when the mean anomaly is given, and this requires the solution of a transcendental equation. Then Eqn. (5-70) can be used to obtain θ. We also find that

$$r = a(1 - \epsilon \cos E) \qquad (5\text{-}73)$$

where we have eliminated θ between Eqns. (5-49) and (5-70).

Hyperbolic orbit. The time since perigee can also be calculated for the case of hyperbolic orbits by using an auxiliary angle H which corresponds roughly to the eccentric anomaly E for elliptical orbits. Without going into the details of the geometry and the derivation we will present the results.

The auxiliary angle H is obtained from

$$\tan H = \frac{r}{b} \sin \Theta = \frac{\sqrt{\epsilon^2 - 1} \sin \Theta}{1 + \epsilon \cos \Theta} \qquad (5\text{-}74)$$

or from

$$\tan \frac{H}{2} = \sqrt{\frac{\epsilon - 1}{\epsilon + 1}} \tan \frac{\Theta}{2} \qquad (5\text{-}75)$$

corresponding to Eqns. (5-69) and (5-70) for the elliptical case. The value of H is limited to the range $-\pi/2 \leqslant H \leqslant \pi/2$ and, of course, Θ is limited to the range $|\Theta| \leqslant (\pi - \phi)$ where ϕ is the asymptote angle as given by Eqn. (5-57).

The time since perigee is

$$t = \sqrt{\frac{a^3}{\mu}} (\epsilon \tan H - \nu) \qquad (5\text{-}76)$$

where

$$\sinh \nu = \tan H$$

or

$$\nu = \ln [\tan (\frac{\pi}{4} + \frac{H}{2})] \qquad (5\text{-}77)$$

Corresponding to Eqn. (5-73) we have

$$r = a (\frac{\epsilon}{\cos H} - 1) \qquad (5\text{-}78)$$

Parabolic orbit. The true anomaly Θ can be calculated as a function of r or $\dot{r}$ by using Eqns. (5-55) and (5-56). First we obtain

$$\cos \Theta = 2(\frac{r_p}{r}) - 1 \qquad (5\text{-}79)$$

Differentiating Eqn. (5-79) with respect to time and using Eqns. (5-34) and (5-63), we also obtain

$$\sin \Theta = \frac{2r_p}{h} \dot{r} = \sqrt{\frac{2r_p}{\mu}} \dot{r} \qquad (5\text{-}80)$$

Other convenient forms are obtained from Eqns. (5-79) and (5-80) by noting that

$$\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \sqrt{\frac{r}{r_p} - 1} \qquad (5\text{-}81a)$$

and also

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{\dot{r}}{r\dot{\theta}} = \tan \gamma \qquad (5\text{-}81b)$$

The time since perigee can be obtained by using Eqns. (5-34), (5-56), and (5-63) to give

$$r^2\dot{\theta} = h = \sqrt{2r_p\mu}$$

and, substituting the basic equation of the parabola as given by Eqn. (5-55), there results

$$\frac{4r_p^2}{(1 + \cos \theta)^2}\,\dot{\theta} = \frac{r_p^2}{\cos^4 \frac{\theta}{2}}\,\dot{\theta} = \sqrt{2r_p\mu}$$

which can be integrated, giving

$$\sqrt{\frac{2r_p^3}{\mu}} \int_0^{\theta} \frac{d\,\frac{\theta}{2}}{\cos^4 \frac{\theta}{2}} = t$$

or

$$t = \sqrt{\frac{2r_p^3}{\mu}}\left(\tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2}\right) \qquad (5\text{-}82a)$$

Of course, an exactly parabolic orbit will not exist in practice. It is desirable to be able to calculate the time of flight for approximately parabolic orbits, i.e., for $\epsilon \approx 1$. This can be accomplished by taking the time of flight equation for the elliptic case and expanding in terms of the deviation δ of the eccentricity from unity. The result for the time since perigee is

$$t \approx \sqrt{\frac{2r_p^3}{\mu}}\left[\left(1 + \frac{\delta}{4}\right) \tan \frac{\theta}{2} + \frac{1}{3}\left(1 - \frac{3}{4}\delta\right) \tan^3 \frac{\theta}{2}\right] \qquad (5\text{-}82b)$$

where

$$\delta = 1 - \epsilon$$

This result is approximate but gives quite good results for $-0.2 < \delta < 0.2$ and $|\theta| < \pi/2$. It avoids the difficulties of calculating E or H in the elliptical or hyperbolic cases, respectively, when $\epsilon \approx 1$ and these angles tend to be very small.

5-8. Satellite Orbits about the Earth

In this section we will be concerned with planetary satellite orbits in which the mass of the orbiting body is negligible compared to that of the planet. In particular we will turn our attention to orbits about the earth.

Circular orbits. We note from Eqn. (5-29) that

$$\mu = Gm_e$$

where m_e is the mass of the earth where we neglect the satellite mass. However, it is more convenient in solving earth satellite problems to obtain the gravitational constant μ in terms of the acceleration of gravity g. At the earth's surface the radial force due to gravity on a unit mass is

$$-g = -\frac{\mu}{R^2}$$

or, solving for μ,

$$\mu = gR^2 \tag{5-83}$$

where a spherical earth is assumed of radius R.

$$R = 3960 \text{ mi} = 20.91 \times 10^6 \text{ ft}$$

$$g = 32.2 \text{ ft/sec}^2 = 7.90 \times 10^4 \text{ mi/hr}^2$$

$$\mu = 1.407 \times 10^{16} \text{ ft}^3/\text{sec}^2 = 1.239 \times 10^{12} \text{ mi}^3/\text{hr}^2$$

Now suppose that a satellite is in a circular orbit at the earth's surface, assuming no atmospheric drag. The gravitational attraction per unit mass will equal the centripetal acceleration.

$$g = \frac{v_c^2}{R}$$

or

$$v_c = \sqrt{gR} \tag{5-84}$$

where v_c is the circular orbit speed at the earth's surface.

$$v_c = 25{,}950 \text{ ft/sec} = 17{,}700 \text{ mi/hr}$$

From Eqns. (5-83) and (5-84) we obtain μ in terms of v_c

$$\mu = Rv_c^2 \tag{5-85}$$

The total energy per unit mass in this reference orbit is

$$e_c = \frac{1}{2} v_c^2 - \frac{\mu}{R}$$

or, using Eqns. (5-84) and (5-85),

$$e_c = -\frac{1}{2} v_c^2 = -\frac{1}{2} gR \tag{5-86}$$

For a general circular orbit, the speed v is found by equating gravitational and centripetal force magnitudes.

$$\frac{v^2}{r} = \frac{\mu}{r^2}$$

which reduces to

$$v = v_c \sqrt{\frac{R}{r}} \tag{5-87}$$

with the aid of Eqn. (5-85). The total energy is found from Eqn. (5-66).

$$e = \frac{1}{2} v^2 - \frac{\mu}{r} = \frac{1}{2} v^2 - \frac{R}{r} v_c^2 \tag{5-88}$$

which simplifies with the aid of (5-85) and (5-87) to the following result for the case of a circular orbit,

$$e = -\frac{R}{2r} v_c^2 \tag{5-89}$$

Escape velocity. The minimum speed at the earth's surface which is necessary in order to escape from the earth is known as the escape velocity v_e. It is the speed that corresponds to a parabolic or zero energy orbit. Setting $e = 0$ and $r = R$ in Eqn. (5-88), we obtain

$$v_e = \sqrt{2}\, v_c \tag{5-90}$$

or, numerically,

$$v_e = 36{,}700 \text{ ft/sec} \doteq 25{,}000 \text{ mi/hr}$$

Elliptical and hyperbolic orbits. Suppose that an earth satellite has a speed v and flight path angle γ at a certain point in its orbit, as shown in Fig. 5-9. (Note that γ is the angle of the velocity vector $\overline{v}$ above the local horizontal, the local vertical being opposite to the direction of the gravitational force at that point.) From a knowledge of r, v, and γ, we can calculate the principal parameters of the orbit.

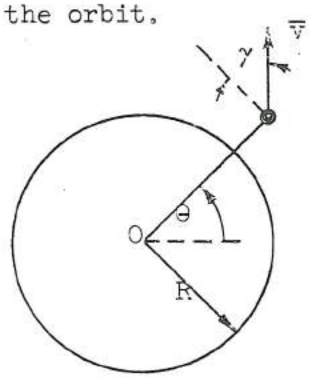

Fig. 5-9

The total energy per unit mass is found from Eqn. (5-88),

$$e = \frac{1}{2} v_c^2 \left[\left(\frac{v}{v_c}\right)^2 - 2\,\frac{R}{r} \right] \tag{5-91}$$

The angular momentum is

$$h = rv \cos \gamma \tag{5-92}$$

From Eqns. (5-65), (5-85), and (5-91) we obtain an expression for the semimajor axis

$$a = \mp \frac{v_c^2}{2e} R = \pm \frac{R}{2 \frac{R}{r} - (\frac{v}{v_c})^2} \tag{5-93}$$

where the upper and lower signs refer to elliptical and hyperbolic orbits, respectively. Note that a is independent of the direction of $\overline{v}$.

The eccentricity is found by substituting expressions for e and h from Eqns. (5-91) and (5-92) into Eqn. (5-62).

$$\epsilon = \sqrt{1 + (\frac{r}{R})^2 (\frac{v}{v_c})^2 \left[(\frac{v}{v_c})^2 - 2 \frac{R}{r} \right] \cos^2\gamma} \tag{5-94}$$

Using Eqn. (5-93), this can be written in the form

$$\epsilon = \sqrt{1 - (\frac{r^2}{aR}) (\frac{v}{v_c})^2 \cos^2\gamma} \tag{5-95}$$

Knowing a and ϵ, the perigee distance can be calculated from Eqn. (5-50) or (5-61).

$$r_p = \pm a(1 - \epsilon) \tag{5-96}$$

Another parameter of importance is the value of the true anomaly θ, or, equivalently, the location of the perigee. For example, for the case of an elliptical orbit, we find from the orbit equation (5-49) that

$$\cos \theta = \frac{1}{\epsilon} \left[\frac{a}{r} (1 - \epsilon^2) - 1 \right]$$

Differentiating with respect to time, we obtain

$$\sin \theta = \frac{1}{\epsilon} \left[\frac{a}{r} (1 - \epsilon^2) \right] \frac{\dot{r}}{r\dot{\theta}}$$

and thus find, by dividing the equations, that

$$\cot \Theta = \cot \gamma \left[1 - \frac{1}{\frac{a}{r}(1 - \epsilon^2)} \right] \tag{5-97}$$

where we see from Fig. 5-9 that

$$\tan \gamma = \frac{\dot{r}}{r\dot{\Theta}} \tag{5-98}$$

But, from Eqns. (5-93) and (5-94)

$$\frac{a}{r}(1 - \epsilon^2) = \left(\frac{r}{R}\right)\left(\frac{v}{v_c}\right)^2 \cos^2\gamma \tag{5-99}$$

so Eqn. (5-97) can be written in the form

$$\cot \Theta = \cot \gamma \left[1 - \frac{1}{\left(\frac{r}{R}\right)\left(\frac{v}{v_c}\right)^2 \cos^2\gamma} \right]$$

or

$$\tan \Theta = \frac{\sin \gamma \cos \gamma}{\cos^2\gamma - \left(\frac{R}{r}\right)\left(\frac{v_c}{v}\right)^2} \tag{5-100}$$

This result is also valid for parabolic or hyperbolic orbits.

The period is found from Eqns. (5-68), (5-85), and (5-93).

$$T = 2\pi \sqrt{\frac{a^3}{\mu}} = \frac{2\pi R/v_c}{\left[2\frac{R}{r} - \left(\frac{v}{v_c}\right)^2 \right]^{3/2}} \tag{5-101}$$

where the numerator is the period T_c of a circular orbit at the earth's surface.

$$T_c = \frac{2\pi R}{v_c} = 1.407 \text{ hr} = 84.4 \text{ min}$$

Finally, it is convenient to obtain an expression for the speed at any point in the orbit as a function of the radial distance r and semimajor axis a. From

Eqn. (5-93) we see that

$$v^2 = v_c^2 \left(2 \frac{R}{r} \mp \frac{R}{a}\right) \qquad (5\text{-}102)$$

The equations have been based upon the speed v_c of a circular orbit at the earth's surface, i.e., at a distance R from the center of the earth. In all cases they could be interpreted as applying equally well to any circular reference orbit at an arbitrary distance R from the center of a large, spherical attracting body such as a planet. In each case the circular speed v_c would be calculated for the reference orbit of radius R.

5.9. Examples

Example 5-2. An earth satellite has a perigee speed of 30,000 ft/sec and an apogee speed of 10,000 ft/sec. Determine r_p, r_a, ϵ, and a.

Since angular momentum is conserved and since $\gamma = 0$ at perigee and apogee, we find from Eqn. (5-92) that

$$30{,}000\ r_p = 10{,}000\ r_a$$

or

$$\frac{r_p}{r_a} = \frac{1}{3}$$

From Eqns. (5-52) and (5-53) we obtain

$$\epsilon = \frac{r_a - r_p}{r_a + r_p} = \frac{1 - r_p/r_a}{1 + r_p/r_a} = 0.50$$

The total energy e is conserved, so we can equate the values computed from Eqn. (5-66) or (5-91) at perigee and apogee.

$$\frac{1}{2}(30{,}000)^2 - \frac{1.407 \times 10^{16}}{r_p} = \frac{1}{2}(10{,}000)^2 - \frac{1.407 \times 10^{16}}{r_a}$$

where the units for each term are ft^2/sec^2. We have already found that $r_a = 3r_p$ so we can solve the above equation for r_p and r_a.

$$r_p = 23.45 \times 10^6 \text{ ft} = 4440 \text{ mi}$$

$$r_a = 70.35 \times 10^6 \text{ ft} = 13{,}320 \text{ mi}$$

Also, from Eqn. (5-52),

$$a = \frac{1}{2}(r_p + r_a) = 46.90 \times 10^6 \text{ ft} = 8880 \text{ mi}$$

Example 5-3. Immediately after the last stage burnout of a rocket, the following data are received concerning its motion:

$$r = 4200 \text{ mi} = 22.18 \times 10^6 \text{ ft}$$

$$\dot{r} = 12{,}000 \text{ ft/sec}$$

$$\dot{\theta} = 1.41 \times 10^{-3} \text{ rad/sec}$$

where the reference frame is an inertial system with its origin at the earth's center. Find the apogee distance r_a and the time of the rocket's return to the earth. (See Fig. 5-10.)

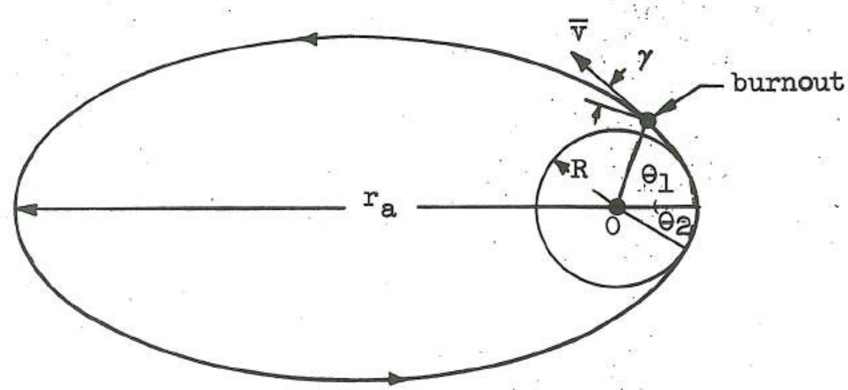

Fig. 5-10

First calculate the initial flight path angle γ using Eqn. (5-98).

$$\tan\gamma = \frac{\dot{r}}{r\dot{\theta}} = \frac{12{,}000}{31{,}270} = 0.384$$

$$\gamma = 21°$$

The burnout velocity is

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = 33{,}500 \text{ ft/sec}$$

Calculating the ratios (at burnout)

$$\frac{v}{v_c} = \frac{33,500}{25,950} = 1.291$$

$$\frac{r}{R} = \frac{4200}{3960} = 1.061$$

we obtain the eccentricity ϵ from Eqn. (5-94).

$$\epsilon = \sqrt{1 - .3578} = 0.801$$

Since $\epsilon < 1$, the orbit is an ellipse. The semimajor axis is obtained from Eqn. (5-93).

$$a = \frac{3960}{0.219} = 18,100 \text{ mi} = 95.5 \times 10^6 \text{ ft}$$

The apogee distance is

$$r_a = a(1 + \epsilon) = 32,600 \text{ mi} = 172 \times 10^6 \text{ ft}$$

In order to calculate the time of flight we need first to calculate the angles Θ_1 and Θ_2. The true anomaly Θ_1 at burnout is found from the basic equation of the ellipse, Eqn. (5-49), or from Eqn. (5-100).

$$\Theta_1 = 47.5°$$

The corresponding eccentric anomaly E is found from Eqn. (5-70).

$$\tan \frac{E_1}{2} = \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \tan \frac{\Theta_1}{2} = .1463$$

and therefore,

$$E_1 = 16.6°$$

The total period for the elliptical orbit is obtained by using Eqn. (5-101)

$$T = \frac{1.407}{(.219)^{3/2}} = 13.73 \text{ hr}$$

Therefore, the time since perigee at burnout is

$$t_1 = \frac{T}{2\pi}(E_1 - \epsilon \sin E_1) = 0.133 \text{ hr}$$

in accordance with Eqn. (5-71).

By a similar process we can calculate Θ_2, the corresponding eccentric anomaly E_2, and also the time before perigee at which the rocket hits the earth. The results are

$$\Theta_2 = 37.5°$$

$$E_2 = 12.9°$$

$$t_2 = 0.101 \text{ hr}$$

The total time of flight from burnout until impact at the earth's surface is

$$T - t_1 - t_2 = 13.50 \text{ hr}$$

SPACE VEHICLE TRAJECTORIES

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SPACE VEHICLE TRAJECTORIES

PREFACE

Space vehicle mechanics leans heavily on the work of the great mathematicians and astronomers of the 18th and 19th centuries, which is presented in many very fine books, such as Whittakers "Treatise on the Analytical Dynamics of Particles and Rigid Bodies." These notes are intended only to serve as an abstract of some of the important subjects, and to show a few applications of the techniques developed. None of the material (except the errors and part of Appendix B) is original with the author.

L. E. Fogarty

June, 1961

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INTRODUCTION - NOTATION AND NOMENCLATURE

A. Kepler's Laws of Planetary Motion

On the basis of observational data gathered in the latter part of the 16th century by Tycho Brahé, his disciple and colleague, Johannes Kepler announced the following three laws of planetary motion.

1. The radius vector of each planet with respect to the sun as the origin sweeps over equal areas in equal times.

2. The orbit of each planet is an ellipse with the sun at one of its foci.

3. The squares of the periods of the planets are to each other as the cubes of the major semi-axes of their respective orbits.

From the three Laws of Planetary Motion, Issac Newton deduced the Law of Universal Gravitation -- every two particles of matter in the universe attract each other with a force which acts in a line joining them, the magnitude of the force being proportional to the product of their masses and inversely proportional to the square of their distance apart.

Except for the effects of relativity, Newton's Laws of Motion and of gravitation apparently can be used to explain completely the motion of the macroscopic celestial bodies of our solar system.

B. Notation for the Ellipse

Because Kepler's Laws furnish a good approximation to the motion, it is natural to adopt notation and nomenclature which apply to orbits which are conic sections, particularly to elliptical orbits.

If a body obeys Kepler's Laws and moves in an elliptical orbit, its position will be specified for all time provided that the size, shape and position of the ellipse in space, and the location of the body on the ellipse at some time, are given.

The size and shape of the ellipse are specified by its elements, usually the semi-latus rectum, p, and the eccentricity, e or, alternatively, the semi-major axis, a, and semi-minor axis, b.

The elements and relations between them are shown in Figure 1.

C. Spatial Orientation of the Ellipse

The orientation of the ellipse in space is specified with respect to a reference plane and a reference line in that plane, as shown in Figure 2. Two reference planes are commonly used; the plane of the ecliptic is the plane of the Earth's orbit about the Sun and the equatorial plane is the plane normal to the Earth's axis of rotation, through the center of the Earth. The line of intersection of these two planes in the direction of the constellation Aries is the commonly used reference direction and is called the vernal equinox, ♈.

If the origin for the reference line is the center of the Earth, the axes are geocentric; if the origin is the center of the Sun, the axes are heliocentric; if the origin is the center of the Moon, the axes are selenocentric.

The line of intersection of the plane of the ellipse and the reference plane is the line of nodes. When the body passes through the line of nodes from South to North, it passes through the ascending node. (Conversely, the descending node is the half of the node line which the body passes through from North to South.)

The longitude of the ascending node, ☊, is the angle subtended at the origin by the vernal equinox and the ascending node.

The angle of inclination, i, is the angle between the plane of the ellipse and the reference plane. It is measured in the plane normal to both, in the counterclockwise direction, looking from the ascending node toward the origin.

The point on the ellipse which is closest to the origin (the nearest apse) is the periapsis. It is called the perigee if the axes are geocentric or the perihelion if the axes are heliocentric. The most distant point from the origin is the apoapsis, apogee or aphelion. The periapsis and the apoapsis are joined by the line of apsides, (which coincides with the major axis of the ellipse). The angle from the ascending node to the periapsis measured in the direction of motion of the body is the argument of periapsis, (perigee or perihelion) and is denoted by the symbol, ω.

D. Position of the Body in the Orbit

The position of the body in its elliptical orbit is specified by the true anomaly, v, the polar angle in the orbit, counted from the periapsis; or by the argument of the latitude, u, the polar angle measured from the ascending node.

$$u = v + \omega$$

The time at which the body passes the periapsis is the <u>epoch</u>, and usually is denoted by T, or τ.

E. Additional Definitions

The quantities which have been defined are sufficient to specify the position in space of a body which obeys Kepler's Laws, for all time. A number of additional quantities are commonly used, among them:

The <u>eccentric anomaly</u>, E, is an auxiliary angle defined by

$$a - r = a\, e \cos E$$

where r is the radius from the origin to the body.

The <u>mean anomaly</u>, M, is the angle which would have been described by the radius vector if it had moved uniformly with the <u>average rate</u>, or <u>mean motion</u>, n.

$$n(t - \tau) = M = E - e \sin E$$